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II.—*On the Theory of Planetary Disturbance.* *By the* Rev. BRICE BRONWIN.

Read November 30, 1848.

1. IN this paper I shall consider, with M. Hansen, the disturbance as affecting the radius vector (or rather the mean distance) and the mean longitude; and I shall first, after his manner, employ two times. Various formulæ not noticed by him (one of them fundamental) are given, in the hope that they may some time be made useful. The principal equations are investigated in a way that leads to some very elegant formulæ, and the elimination of the quantities containing both the times is effected in a very simple manner. In finding the latitude, I propose to introduce the latitude itself and the reduction into the disturbance function; by which means the part of that function depending on the inclination of the orbit to the fixed plane is greatly simplified, and the determination of the latitude and reduction rendered easier. But it is not my intention in the present paper to develope the functions in series of sines or cosines.

The well-known differential equations of a planet's motion, referred to the plane of the orbit, are

$$\left.\begin{aligned} \frac{d^2r}{dt^2}-\frac{h^2}{r^3}+\frac{\mu}{r^2}+\frac{dR}{dr}=0, \\ r^2\frac{dv}{dt}=h=h_0-\int\frac{dR}{dv}\,dt, \end{aligned}\right\} \tag{1}$$

$$R=\frac{m'r}{r'^2}\cos\chi-\frac{m'}{(r'^2-2r'r\cos\chi+r^2)^{\frac{1}{2}}};\ \cos\chi=\frac{xx'+yy'+zz'}{r'r};$$

where x, y, and z are the rectangular co-ordinates of the disturbed ($z=0$), and x', y', and z' those of the disturbing body. The meaning of the other symbols is obvious.

When $m'=0$, let the integrals of (1) be

$$r_0=f\,(t,\,a_0,\,e_0,\,\pi_0,\,\epsilon_0),\quad v_0=\phi\,(t,\,a_0,\,e_0,\,\pi_0,\,\epsilon_0).$$

And when the disturbing force is restored, let

$$r=f\,(t,\,a,\,e,\,\pi,\,\epsilon),\quad v=\phi\,(t,\,a,\,e,\,\pi,\,\epsilon).$$

Thus r and v are the same functions of t as when $m'=0$, the elements a, e, &c., being variable, and determined in the usual manner. And, τ being a new time, we make

$$\rho=f\,(\tau,\,a,\,e,\,\pi,\,\epsilon),\quad \lambda=\phi\,(\tau,\,a,\,e,\,\pi,\,\epsilon).$$

We further suppose,

$$r_{,}=f\,(z,\,a_{,},\,e_{,},\,\pi_{,},\,\epsilon_{,}),\quad v_{,}=\phi\,(z,\,a_{,},\,e_{,},\,\pi_{,},\,\epsilon_{,})\,;$$
$$\rho_{,}=f\,(\zeta,\,a_{,},\,e_{,},\,\pi_{,},\,\epsilon_{,}),\quad \lambda_{,}=\phi\,(\zeta,\,a_{,},\,e_{,},\,\pi_{,},\,\epsilon_{,})\,;$$

the elements $a_{,}$, $e_{,}$, &c., and $h_{,}$, to be presently introduced, being constants, z a function of t, and ζ a function of τ and t. To these we must add the assumed relations,

$$r=r_{,}\beta,\quad \rho=\rho_{,}\xi,\quad v=v_{,}+\mathfrak{E}nt,\quad \lambda=\lambda_{,}+\mathfrak{E}nt.$$

By changing τ into t, we change ρ into r, λ into v, $\rho_{,}$ into $r_{,}$, $\lambda_{,}$ into $v_{,}$, ξ into β, and ζ into z.

From what has been given above, we have necessarily $r_{,}^2\dfrac{dv_{,}}{dz}=h_{,}$, $\rho_{,}^2\dfrac{d\lambda}{d\zeta}=h_{,}$. And from the way in which a, e, &c., are found, $\delta r=0$, $\delta v=0$, the characteristic δ denoting the variation relative to a, e, &c.; therefore, $\rho^2\dfrac{d\lambda}{d\tau}=h$.

But $\rho^2\dfrac{d\lambda}{d\tau}=\rho_{,}^2\xi^2\dfrac{d\lambda_{,}}{d\tau}=\rho_{,}^2\xi^2\dfrac{d\lambda_{,}}{d\zeta}\dfrac{d\zeta}{d\tau}=h_{,}\xi^2\dfrac{d\zeta}{d\tau}$. Consequently $h_{,}\xi^2\dfrac{d\zeta}{d\tau}=h$; or,

$$\frac{d\zeta}{d\tau}=\frac{h}{h_{,}\xi^2}. \tag{2}$$

Also $r^2\dfrac{dv}{dt}=r^2\dfrac{dv_{,}}{dt}+\mathfrak{E}nr^2=r_{,}^2\beta^2\dfrac{dv_{,}}{dz}\dfrac{dz}{dt}+\mathfrak{E}nr_{,}^2\beta^2=h_{,}\beta^2\dfrac{dz}{dt}+\mathfrak{E}nr_{,}^2\beta^2=h$; or,

$$\frac{dz}{dt}=\frac{h}{h_{,}\beta^2}-\frac{\mathfrak{E}nr_{,}^2}{h_{,}}. \tag{3}$$

The two theorems now found are fundamental. The latter is not noticed by M. Hansen; the former, or one equivalent to it, is employed by him. But there is another fundamental theorem which he has not found; for we have necessarily

$$\frac{d^2\rho}{d\tau^2}-\frac{h^2}{\rho^3}+\frac{\mu}{\rho^2}=0,$$

as in an undisturbed orbit; because, since d operates on τ only, the elements and h may vary in any manner whatever, and this equation will still hold.

Now $\frac{d\rho}{d\tau}=\rho_{,}\frac{d\xi}{d\tau}+\xi\,\frac{d\rho_{,}}{d\tau}=\rho_{,}\frac{d\xi}{d\tau}+\xi\,\frac{d\rho_{,}}{d\zeta}\frac{d\zeta}{d\tau}=\rho_{,}\frac{d\xi}{d\tau}+\frac{h}{h_{,}\xi}\frac{d\rho_{,}}{d\zeta}$, by (2); and $\frac{d^2\rho}{d\tau^2}=\rho_{,}\frac{d^2\xi}{d\tau^2}+\frac{d\rho_{,}}{d\zeta}\frac{d\zeta}{d\tau}\frac{d\xi}{d\tau}-\frac{h}{h_{,}\xi^2}\frac{d\rho_{,}}{d\zeta}\frac{d\xi}{d\tau}+\frac{h}{h_{,}\xi}\frac{d^2\rho_{,}}{d\zeta^2}\frac{d\zeta}{d\tau}=\rho_{,}\frac{d^2\xi}{d\tau^2}+\frac{h}{h_{,}\xi}\frac{d^2\rho_{,}}{d\zeta^2}\frac{d\zeta}{d\tau}$, by (2); $=\rho_{,}\frac{d^2\xi}{d\tau^2}+\frac{h^2}{h_{,}^2\xi^3}\,\frac{d^2\rho_{,}}{d\zeta^2}$, by the same.

But $\frac{d^2\rho_{,}}{d\zeta^2}-\frac{h_{,}^2}{\rho_{,}^3}+\frac{\mu}{\rho_{,}^2}=0$. Therefore, substituting for $\frac{d^2\rho}{d\tau^2}$ and $\frac{d^2\rho_{,}}{d\zeta^2}$ their values, $\frac{h^2}{\rho^3}-\frac{\mu}{\rho^2}$ and $\frac{h_{,}^2}{\rho_{,}^3}-\frac{\mu}{\rho_{,}^2}$, we have

$$\frac{h^2}{\rho^3}-\frac{\mu}{\rho^2}=\rho_{,}\frac{d^2\xi}{d\tau^2}+\frac{h^2}{h_{,}^2\xi^3}\left(\frac{h_{,}^2}{\rho_{,}^3}-\frac{\mu}{\rho_{,}^2}\right);$$

or, since $\epsilon^3\rho_{,}^3=\rho^3$,

$$-\frac{\mu}{\rho^2}=\rho_{,}\frac{d^2\xi}{d\tau^2}-\frac{h^2}{h_{,}^2\xi^3}\,\frac{\mu}{\rho_{,}^2};$$

which easily reduces to

$$\frac{d^2\xi}{d\tau^2}=\frac{\mu}{\rho^3}\left(\frac{h^2}{h_{,}^2}-\xi\right). \qquad (4)$$

There are still two other theorems to be investigated, which are given by M. Hansen; but found by him in a manner very different from that which I shall employ.

$$\frac{1}{r}=\frac{\mu}{h^2}+\frac{\mu e}{h^2}\cos\,(v-\pi)\text{ gives }\frac{1}{r}-\frac{\mu}{h^2}=\frac{\mu e}{h^2}\cos\,(v-\pi)=$$

$$\frac{\mu e}{h^2}\,(\cos\pi\cos v+\sin\pi\sin v).$$

And differentiating relative to t,

$$\frac{1}{r^2}\frac{dr}{dt}=\frac{\mu e}{h^2}\sin(v-\pi)\frac{dv}{dt}\text{; or, since } r^2\frac{dv}{dt}=h,$$

$$\frac{1}{h}\frac{dr}{dt}=\frac{\mu e}{h^2}\sin(v-\pi)=\frac{\mu e}{h^2}(\cos\pi\sin v-\sin\pi\cos v).$$

Whence we easily deduce

$$\left(\frac{1}{r}-\frac{\mu}{h^2}\right)\cos v+\frac{1}{h}\frac{dr}{dt}\sin v=\frac{\mu e}{h^2}\cos\pi.$$

$$\left(\frac{1}{r}-\frac{\mu}{h^2}\right)\sin v-\frac{1}{h}\frac{dr}{dt}\cos v=\frac{\mu e}{h^2}\sin\pi.$$

In like manner,

$$\left(\frac{1}{\rho}-\frac{\mu}{h^2}\right)\cos\lambda+\frac{1}{h}\frac{d\rho}{d\tau}\sin\lambda=\frac{\mu e}{h^2}\cos\pi.$$

$$\left(\frac{1}{\rho}-\frac{\mu}{h^2}\right)\sin\lambda-\frac{1}{h}\frac{d\rho}{d\tau}\cos\lambda=\frac{\mu e}{h^2}\sin\pi.$$

Equating the two sets of values of $\frac{\mu e}{h^2}\cos\pi$ and $\frac{\mu e}{h^2}\sin\pi$,

$$\left.\begin{aligned}\left(\frac{1}{r}-\frac{\mu}{h^2}\right)\cos v+\frac{1}{h}\frac{dr}{dt}\sin v&=\left(\frac{1}{\rho}-\frac{\mu}{h^2}\right)\cos\lambda+\frac{1}{h}\frac{d\rho}{d\tau}\sin\lambda,\\ \left(\frac{1}{r}-\frac{\mu}{h^2}\right)\sin v-\frac{1}{h}\frac{dr}{dt}\cos v&=\left(\frac{1}{\rho}-\frac{\mu}{h^2}\right)\sin\lambda-\frac{1}{h}\frac{d\rho}{d\tau}\cos\lambda.\end{aligned}\right\}\quad(5)$$

Multiplying these by $\cos v$, $\sin v$, $\cos\lambda$, and $\sin\lambda$, and adding and subtracting results,

$$\left.\begin{aligned}\frac{1}{r}-\frac{\mu}{h^2}&=\left(\frac{1}{\rho}-\frac{\mu}{h^2}\right)\cos(\lambda-v)+\frac{1}{h}\frac{d\rho}{d\tau}\sin(\lambda-v),\\ \frac{1}{h}\frac{dr}{dt}&=-\left(\frac{1}{\rho}-\frac{\mu}{h^2}\right)\sin(\lambda-v)+\frac{1}{h}\frac{d\rho}{d\tau}\cos(\lambda-v),\\ \frac{1}{\rho}-\frac{\mu}{h^2}&=\left(\frac{1}{r}-\frac{\mu}{h^2}\right)\cos(\lambda-v)-\frac{1}{h}\frac{dr}{dt}\sin(\lambda-v),\\ \frac{1}{h}\frac{d\rho}{d\tau}&=\left(\frac{1}{r}-\frac{\mu}{h^2}\right)\sin(\lambda-v)+\frac{1}{h}\frac{dr}{dt}\cos(\lambda-v).\end{aligned}\right\}\quad(6)$$

The equations (5) may be put under the following form :

$$\frac{d}{dt}(r\sin v) - \frac{d}{d\tau}(\rho\sin\lambda) = \frac{\mu}{h}(\cos v - \cos\lambda).$$

$$\frac{d}{dt}(r\cos v) - \frac{d}{d\tau}(\rho\cos\lambda) = \frac{\mu}{h}(\sin\lambda - \sin v).$$

The first and second of (6) may be written thus:

$$\frac{1}{r} = \frac{\mu}{h^2}\{1-\cos(\lambda-v)\} + \frac{1}{h}\frac{d}{d\tau}\{\rho\sin(\lambda-v)\};$$

$$\frac{dr}{dt} = \frac{\mu}{h}\sin(\lambda-v) + \frac{d}{d\tau}\{\rho\cos(\lambda-v)\}.$$

These, and the third and fourth of (6), are very elegant equations, and are deserving of notice on the ground that, possibly, some use may be made of them. I shall only employ the third of (6). Differentiating it relative to t, there results,

$$\frac{1}{\rho^2}\frac{d\rho}{dt} = \left\{\frac{2\mu}{h^3} - \frac{2\mu}{h^3}\cos(\lambda-v)\right\}\frac{dh}{dt} + \left\{\frac{1}{h}\frac{d\lambda}{dt}\cos(\lambda-v) - \frac{1}{h^2}\frac{dh}{dt}\sin(\lambda-v)\right\}\frac{dr}{dt}$$
$$+\left(\frac{1}{r} - \frac{\mu}{h^2}\right)\frac{d\lambda}{dt}\sin(\lambda-v) + \left(\frac{1}{h}\frac{d^2r}{dt^2} - \frac{h}{r^3} + \frac{\mu}{hr^2}\right)\sin(\lambda-v).$$

Eliminating $\frac{dr}{dt}$ by the third and fourth of (6), and substituting for $\frac{d^2r}{dt^2} - \frac{h^2}{r^3} + \frac{\mu}{r^2}$ its value from (1), we find

$$\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{1}{h}\frac{d\rho}{d\tau}\frac{d\lambda}{dt} + \left\{\frac{1}{\rho} + \frac{\mu}{h^2} - \left(\frac{1}{r} + \frac{\mu}{h^2}\right)\cos(\lambda-v)\right\}\frac{dh}{hdt} - \frac{1}{h}\sin(\lambda-v)\frac{dR}{dr}.$$

Multiply this by h, put for $\frac{dh}{dt}$ its value from (1), and for $\frac{h}{\rho^2}$ in the first member its value $\frac{d\lambda}{d\tau}$; then making

$$T = \left\{\left(\frac{1}{r} + \frac{\mu}{h^2}\right)\cos(\lambda-v) - \left(\frac{1}{\rho} + \frac{\mu}{h^2}\right)\right\}\frac{dR}{dv} - \sin(\lambda-v)\frac{dR}{dr},$$

we have

$$\frac{d\rho}{dt}\,\frac{d\lambda}{d\tau}-\frac{d\rho}{d\tau}\,\frac{d\lambda}{dt}=T, \tag{7}$$

which is a very remarkable equation.

From this we deduce successively,

$$\frac{d\rho}{dt}\,\frac{d\lambda_{\prime}}{d\tau}-\frac{d\rho}{d\tau}\,\frac{d\lambda}{dt}=T+\varepsilon n\,\frac{d\rho}{d\tau},$$

$$\frac{h_{\prime}}{\rho_{\prime}^{2}}\left(\frac{d\rho}{dt}\,\frac{d\zeta}{d\tau}-\frac{d\rho}{d\tau}\,\frac{d\zeta}{dt}\right)=T+\varepsilon n\,\frac{d\rho}{d\tau};$$

but

$$\frac{d\rho}{dt}=\rho_{\prime}\frac{d\xi}{dt}+\xi\,\frac{d\rho_{\prime}}{d\zeta}\,\frac{d\zeta}{dt},\ \frac{d\rho}{d\tau}=\rho_{\prime}\frac{d\xi}{d\tau}+\xi\,\frac{d\rho_{\prime}}{d\zeta}\,\frac{d\zeta}{d\tau}.$$

Therefore, by substitution in the above,

$$\frac{h_{\prime}}{\rho_{\prime}}\left(\frac{d\xi}{dt}\,\frac{d\zeta}{d\tau}-\frac{d\xi}{d\tau}\,\frac{d\zeta}{dt}\right)=T+\varepsilon n\,\frac{d\rho}{d\tau}. \tag{8}$$

This again is a very remarkable equation, and has a remarkable correspondence with (7). Dividing it by $\frac{h_{\prime}}{\rho_{\prime}}\frac{d\zeta}{d\tau}=\frac{h}{\rho_{\prime}\xi^{2}}=\frac{h}{\rho\xi}$, we have

$$\frac{d\xi}{dt}-\frac{d\xi}{d\tau}\left(\frac{\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}\right)=\frac{1}{h}\,\rho\xi T+\frac{\varepsilon n\xi}{h}\,\rho\,\frac{d\rho}{d\tau};$$

or,

$$\frac{d\xi}{dt}=\frac{1}{h}\,\rho\xi T+\frac{\varepsilon n\xi}{2h}\,\frac{d(\rho^{2})}{d\tau}+\frac{d\xi}{d\tau}\left(\frac{\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}\right) \tag{9}$$

Equation (7) multiplied by 2ρ gives

$$2\rho\,\frac{d\rho}{dt}\,\frac{d\lambda}{d\tau}-2\rho\,\frac{d\rho}{d\tau}\,\frac{d\lambda}{dt}=2\rho T.$$

But differentiating $\rho^{2}\,\frac{d\lambda}{d\tau}=h$, relative to t, we have

$$\rho^{2}\,\frac{d^{2}\lambda}{d\tau dt}+2\rho\,\frac{d\rho}{dt}\,\frac{d\lambda}{d\tau}=\frac{dh}{dt}.$$

The above, subtracted from this, gives

$$\rho^2 \frac{d^2\lambda}{d\tau\, dt} + 2\rho \frac{d\rho}{d\tau} \frac{d\lambda}{dt} = \frac{dh}{dt} - 2\rho T,$$

or,

$$\frac{d}{d\tau}\left(\rho^2 \frac{d\lambda}{dt}\right) = \frac{dh}{dt} - 2\rho T. \tag{10}$$

This is singular, and very worthy of notice. This theory, as we have treated it, gives rise to many very interesting formulæ.

But $\rho^2 \dfrac{d\lambda}{dt} = \rho^2 \dfrac{d\lambda_{,}}{dt} + \epsilon n \rho^2 = \xi^2 \rho^2 \dfrac{d\lambda_{,}}{d\zeta} \dfrac{d\zeta}{dt} + \epsilon n \rho^2 = h_{,}\xi^2 \dfrac{d\zeta}{dt} + \epsilon n \rho^2$.

Therefore (10) becomes

$$\frac{d}{d\tau}\left(h_{,}\xi^2 \frac{d\zeta}{dt}\right) + \epsilon n \frac{d(\rho^2)}{d\tau} = \frac{dh}{dt} - 2\rho T.$$

Or, since $h_{,}\xi^2 = \dfrac{h}{\frac{d\zeta}{d\tau}}$,

$$\frac{d}{d\tau}\left(\frac{\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}\right) = \frac{1}{h}\frac{dh}{dt} - \frac{1}{h} 2\rho T - \frac{\epsilon n}{h} \frac{d(\rho^2)}{d\tau}. \tag{11}$$

2. In the preceding section we have found all the fundamental formulæ, and indeed more than are absolutely required. We now proceed to discuss some points preparatory to integration.

We might immediately integrate (11) relative to t; but the integral of the second member would contain a very great number of terms with τ in their coefficients, or of the form $\tau f(t)$. We must, therefore, proceed otherwise.

Make $z = t + \omega$, $\zeta = \tau + \phi$. Then $\dfrac{d\zeta}{dt} = \dfrac{d\phi}{dt}$, $\dfrac{d\zeta}{d\tau} = 1 + \dfrac{d\phi}{d\tau}$. Substituting these values in (9) and (11), neglecting terms involving the fourth power of the disturbing force, and putting

$$\frac{1}{h}\rho\xi T + \frac{\epsilon n \xi}{2h} \frac{d(\rho^2)}{d\tau} = P, \; \frac{1}{h}\frac{dh}{dt} - \frac{1}{h} 2\rho T - \frac{\epsilon n}{h}\frac{d(\rho^2)}{d\tau} = S,$$

they become

$$\left.\begin{aligned}\frac{d\xi}{dt} &= P + \frac{d\xi}{d\tau}\frac{d\phi}{dt} - \frac{d\xi}{d\tau}\frac{d\phi}{dt}\frac{d\phi}{d\tau},\\ \frac{d^2\phi}{d\tau dt} &= S + \frac{d}{d\tau}\left(\frac{d\phi}{dt}\frac{d\phi}{d\tau} - \frac{d\phi}{dt}\frac{d\phi^2}{d\tau^2}\right).\end{aligned}\right\} \quad (1)$$

Let the integrals of these relative to t be

$$\xi = f(\tau) + \psi(\tau,t),\ \frac{d\phi}{d\tau} = f_{,}(\tau) + \phi(\tau,t):$$

$f(\tau)$ and $f_{,}(\tau)$ being the arbitraries of the integration. If we make $m' = 0$; then $\epsilon = 0$, and $\lambda = \lambda_{,}$. But in this case λ is a function of τ without t. Consequently, $\lambda_{,}$, ζ, and ϕ, are functions of τ without t; and, therefore, $\frac{d\phi}{dt} = 0$. Hence, all the terms in the second members of (1) vanish, and we have

$$\xi_0 = f(\tau),\ \frac{d\phi_0}{d\tau} = f_{,}(\tau);$$

ξ_0 and $\frac{d\phi_0}{d\tau}$ being the values of ξ and $\frac{d\phi}{d\tau}$ when $m' = 0$.

Now if we make the disturbing force to vanish in (2) and (4) of the first section, they become

$$\left.\begin{aligned}\frac{d\zeta_0}{d\tau} &= \frac{h_0}{h_{,}\xi_0^2},\\ \frac{d^2\xi_0}{d\tau^2} &= \frac{\mu}{\rho_0^3}\left(\frac{h_0^2}{h_{,}^2} - \xi_0\right),\\ 1 + \frac{d\phi_0}{d\tau} &= \frac{h_0}{h_{,}\xi_0^2};\end{aligned}\right\} \quad (2)$$

the last of these being derived from the first by putting for ζ_0 its value. These equations, being integrated, give us ζ_0 and ξ_0, or ξ_0 and ϕ_0.

To integrate the second of these, make $\xi_0 = \frac{h_0^2}{h_{,}^2}(1+\gamma)$, when it becomes

$$\frac{d^2\gamma}{d\tau^2} + \frac{\mu}{\rho_0^3}\gamma = 0, \text{ or } \frac{\rho_0^3}{a_0^3}\frac{d^2\gamma}{d\tau^2} + n_0^2\gamma = 0.$$

To abridge, make $\frac{\rho_0^3}{a_0^3} = 1 + eE_1 + e^2E_2 + \&c.$, where E_{1}, E_2, &c., are known functions of the cosine of $n_0\tau + \epsilon_0 - \pi_0$ and its multiples; and assume

$$\gamma = B + eB_1 + e^2B_2 + \dots .$$

By substitution, and equalling separately to nothing the terms multiplied by the different powers of e, we have

$$\frac{d^2B}{d\tau^2} + n_0^2B = 0, \quad \frac{d^2B_1}{d\tau^2} + n_0^2B_1 + E_1\frac{d^2B}{d\tau^2} = 0, \text{ \&c.}$$

The first of these gives $B = c \cos (n_0\tau + k)$. This value substituted in the second, it will give P_1; and so on. But in reducing the periodicals to one argument, we should have a series of sines and another of cosines, unless we make $k = \epsilon_0 - \pi_0$. In this case we must have $\epsilon_0 - \pi_0 = \epsilon_{,} - \pi_{,}$, and making $n_0 = n_{,}(1 + b)$, we must develope thus:

$$\cos (n_0\tau + \epsilon_0 - \pi_0) = \cos (n_{,}\tau + \epsilon_0 - \pi_0) - bn_{,}\tau \sin (n_{,}\tau + \epsilon_0 - \pi_0) - \text{\&c.}$$

The use of this is, by suitably determining b, to take away an improper term. In some cases, perhaps, both sines and cosines may be necessary.

But perhaps this will be more conveniently done thus:

$$\frac{1}{\rho_0} = \frac{\mu}{h_0^2}\{1 + e_0 \cos (\lambda_0 - \pi_0)\}, \quad \frac{1}{\rho_{,,0}} = \frac{\mu}{h_{,}^2}\{1 + e_{,} \cos (\lambda_{,,0} - \pi_{,})\}.$$

But $\lambda_{,,0} = \lambda_0$. Therefore we must have $\pi_0 = \pi_{,}$, or we should have both sines and cosines. And then we shall have

$$\left(\frac{h_0^2}{\mu\rho_0} - 1\right)\frac{1}{e_0} = \cos (\lambda_0 - \pi_0) = \cos (\lambda_{,,0} - \pi_{,}) = \left(\frac{h_{,}^2}{\mu\rho_{,,0}} - 1\right)\frac{1}{e_{,}};$$

or, after a little reduction,

$$\frac{h_0^2}{\mu\rho_0} = 1 - \frac{e_0}{e_{,}} - \frac{h_{,}^2}{\mu\rho_{,,0}}\,\frac{e_0}{e_{,}}.$$

But $\rho_0 = \rho_{,,0}\xi_0$. Substituting this value, we easily find

$$\frac{1}{\xi_0} = \left(1 - \frac{e_0}{e_{,}}\right)\frac{\mu}{h_0^2}\,\rho_{,,0} + \frac{h_{,}^2}{h_0^2}\,\frac{e_0}{e_{,}};$$

or,

$$\frac{1}{\xi_0} = \frac{e_{,} - e_0}{1 - e_0^2}\,\frac{\rho_{,,0}}{a_0e_{,}} + \frac{a_{,}e_0(1 - e_{,}^2)}{a_0e_{,}(1 - e_0^2)}.$$

Make $e_0 = e_{,}(1 - \alpha)$, α being a small quantity of the order of the disturbing force; and we may write

$$\frac{1}{\xi_0} = \frac{a_{,}}{a_0} A + \alpha B \frac{\rho_{,,0}}{a_0}.$$

To which we may add from the third of (2);

$$\frac{d\phi_0}{d\tau} = \frac{a_0^{\frac{1}{2}}}{a_{,}^{\frac{1}{2}}} \frac{C}{\xi_0^2} - 1.$$

A, B, and C being functions of $e_{,}^2$ and α easily found. But $\rho_{,,0}$, a function of λ_0, and, therefore, a function of $n_0\tau$, must be converted into a function of $n_{,}\tau$ by Taylor's theorem. The results obtained are much more simple than those obtained by M. Hansen in the same case, and I think also more convenient.

If we wish to make $\pi_0 = \pi_{,} + \eta$, we have

$$\cos(\lambda_0 - \pi_0) = \cos\eta \cos(\lambda_0 - \pi_0) + \sin\eta \sin(\lambda_0 - \pi_0),$$

$$\frac{h_0}{\mu e_0} \frac{d\rho_0}{d\tau} = \sin(\lambda_0 - \pi_0) = \cos\eta \sin(\lambda_0 - \pi_{,}) - \sin\eta \cos(\lambda_0 - \pi_{,}),$$

$$\cos(\lambda_{,,0} - \pi_{,}) = \cos(\lambda_0 - \pi_{,}) = \left(\frac{h_{,}^2}{\mu\rho_{,,0}} - 1\right)\frac{1}{e_{,}}.$$

Between these we may eliminate $\sin(\lambda_0 - \pi_{,})$ and $\cos(\lambda_0 - \pi_{,,})$ and obtain a result involving $\frac{d\rho_0}{d\tau}$; but I shall not pursue the subject further.

We may satisfy (2), and all the requisite conditions, by simply making $e_0 = e_{,}$, $\pi_0 = \pi_{,}$, $\xi_0 = \frac{a_0}{a_{,}}$, and $n_{,}\zeta_0 = n_0\tau$; but this would not leave us any arbitrary constant except a_0, and we might have an unsuitable term which we could not get rid of.

We must now proceed to another class of formulæ, some of which will be wanted. But I shall not confine myself to these.

Let $\overline{\left(\frac{d\xi}{d\tau}\right)}$, $\overline{\left(\frac{d\xi}{dt}\right)}$ denote that τ is to be changed into t in these quantities, after the operation of differentiation is performed. And thus this change will be denoted in other cases. Thus from (2) of the first section we have

$$\left.\begin{aligned}\overline{\left(\frac{d\zeta}{d\tau}\right)} &= \frac{h}{h_{,}\beta^2},\\ \overline{\left(\frac{d\phi}{d\tau}\right)} &= \frac{h}{h_{,}\beta^2} - 1.\end{aligned}\right\} \qquad (3)$$

Eliminating $\frac{h}{h_{,}\beta^2}$ between the first of these and (3) of the first section, there results

or,

$$\left.\begin{aligned}\overline{\left(\frac{d\zeta}{d\tau}\right)} &= \frac{dz}{dt} + \frac{\mathfrak{e}nr_{,}^2}{h_{,}};\\ \overline{\left(\frac{d\phi}{d\tau}\right)} &= \frac{dw}{dt} + \frac{\mathfrak{e}nr_{,}^2}{h_{,}}.\end{aligned}\right\} \qquad (4)$$

But it is obvious that $\overline{\left(\frac{d\phi}{d\tau}\right)} + \overline{\left(\frac{d\phi}{dt}\right)} = \frac{dw}{dt}$. Therefore,

$$\overline{\left(\frac{d\phi}{dt}\right)} = -\frac{\mathfrak{e}nr_{,}^2}{h_{,}}. \qquad (5)$$

Also, $\frac{d\rho}{dt} = \rho\frac{d\xi}{dt} + \xi\frac{d\rho}{d\zeta}\frac{d\zeta}{dt}$. Whence

$$\overline{\left(\frac{d\rho}{dt}\right)} = r_{,}\overline{\left(\frac{d\xi}{dt}\right)} + \beta\frac{dr_{,}}{dz}\overline{\left(\frac{d\zeta}{dt}\right)}.$$

But $\overline{\left(\frac{d\rho}{dt}\right)} = \frac{\delta r}{dt} = 0$, from the manner in which the variable elements are found; and $\frac{d\zeta}{dt} = \frac{d\phi}{dt}$. Therefore,

$$r_{,}\overline{\left(\frac{d\xi}{dt}\right)} + \beta\frac{dr_{,}}{dz}\overline{\left(\frac{d\phi}{dt}\right)} = 0,$$

which by (5) gives

$$\overline{\left(\frac{d\xi}{dt}\right)} = \frac{\mathfrak{e}nr}{h_{,}}\frac{dr_{,}}{dz}; \qquad (6)$$

and since also we necessarily have

$$\overline{\left(\frac{d\xi}{d\tau}\right)} + \overline{\left(\frac{d\xi}{dt}\right)} = \frac{d\beta}{dt};$$

therefore,

$$\overline{\left(\frac{d\xi}{d\tau}\right)} = \frac{d\beta}{dt} - \frac{\epsilon nr}{h_{,}}\frac{dr_{,}}{dz}. \tag{7}$$

The preceding are all the formulæ of this kind that are really wanted; but I shall give some others, on the ground that they may possibly be some time found useful. If we could change τ into t in higher differentials than the first, we might, perhaps, by differentiating the fundamental theorems, or any of the others, relative to τ or t, and then changing τ into t, find some new theorems which might be made available for simplification, or some way useful.

From (2), section (1), or $\frac{d\phi}{d\tau} = \frac{h}{h_{,}\xi^2} - 1$, putting $\tau + \phi$ for ζ, we have $\frac{d^2\phi}{d\tau^2} = -\frac{2h}{h_{,}\xi^3}\frac{d\xi}{d\tau}$. Therefore,

$$\overline{\left(\frac{d^2\phi}{d\tau^2}\right)} = -\frac{2h}{h_{,}\beta^3}\overline{\left(\frac{d\xi}{d\tau}\right)};$$

or by (7),

$$\overline{\left(\frac{d^2\phi}{d\tau^2}\right)} = -\frac{2h}{h_{,}\beta^3}\frac{d\beta}{dt} + \frac{\epsilon nh}{h_{,}^2\beta^2}\frac{d(r_{,}^2)}{dz}. \tag{8}$$

Equation (4), section (1), gives immediately

$$\overline{\left(\frac{d^2\xi}{d\tau^2}\right)} = \frac{\mu}{r^3}\left(\frac{h^2}{h_{,}^2} - \beta\right). \tag{9}$$

And from (2), referred to above, after putting $\tau + \phi$ for ζ, and differentiating relative to t, we have

$$\frac{d^2\phi}{d\tau dt} = \frac{1}{h_{,}\xi^2}\frac{dh}{dt} - \frac{2h}{h_{,}\xi^3}\frac{d\xi}{dt},$$

$$\overline{\left(\frac{d^2\phi}{d\tau dt}\right)} = \frac{1}{h_{,}\beta^2}\frac{dh}{dt} - \frac{2h}{h_{,}\beta^3}\overline{\left(\frac{d\xi}{dt}\right)};$$

or, by substitution from (6),

$$\overline{\left(\frac{d^2\phi}{d\tau dt}\right)} = \frac{1}{h_{,}\beta^2}\frac{dh}{dt} - \frac{\epsilon nh}{h_{,}^2\beta^2}\frac{d(r_{,}^2)}{dz}. \tag{10}$$

I might extend this list, but not without some difficulty; and as I am not sure of its utility, I shall here leave it.

We now turn to the equations (1) of this section. The first contains both ξ and ϕ; the second only ϕ. This last, therefore, is preferable; and as we do not want them both, we shall take this, which reduces to

$$\frac{d^2\phi}{d\tau dt} = S + \frac{d^2\phi}{d\tau dt}\left(\frac{d\phi}{d\tau} - \frac{d\phi^2}{d\tau^2}\right) + \frac{d\phi}{dt}\,\frac{d^2\phi}{d\tau^2}\left(1 - 2\,\frac{d\phi}{d\tau}\right).$$

For the first and second power of the disturbing force only,

$$\frac{d^2\phi}{d\tau dt} = S + \frac{d\phi}{d\tau}\,\frac{d^2\phi}{d\tau dt} + \frac{d\phi}{dt}\,\frac{d^2\phi}{d\tau^2} = S + S\,\frac{d\phi}{d\tau} + \frac{d\phi}{dt}\,\frac{d^2\phi}{d\tau^2}.$$

Putting this value in the second member of the above, we have

$$\left.\begin{aligned} \frac{d^2\phi}{d\tau dt} &= S + S\,\frac{d\phi}{d\tau} + \frac{d\phi}{dt}\,\frac{d^2\phi}{d\tau^2} - \frac{d\phi}{dt}\,\frac{d\phi}{d\tau}\,\frac{d^2\phi}{d\tau^2}. \\ S &= \left\{1 + \frac{2\mu\rho}{h^2} - \left(\frac{2}{r} + \frac{2\mu}{h^2}\right)\rho\cos(\lambda - v)\right\}\frac{1}{h}\frac{dR}{dv} + \sin(\lambda - v)\frac{2\rho}{h}\frac{dR}{dr} - \frac{\epsilon n}{h}\frac{d(\rho^2)}{d\tau} \end{aligned}\right\} (11)$$

3. The values of ϕ and ξ would be very troublesome to find; for they would contain a great number of terms having $\tau - t$ in their coefficients, and which in w and β would vanish; and many which, when τ is changed into t, would unite with others. If, therefore, we could get rid of these quantities, we should greatly diminish the labour of integration. This, happily, we are able to effect. To accomplish it we will develope S relative to these quantities.

Equation (2) of section (1) gives $\xi = \frac{h^{\frac{1}{2}}}{h_{,}^{\frac{1}{2}}}\left(1 + \frac{d\phi}{d\tau}\right)^{-\frac{1}{2}} = \frac{h^{\frac{1}{2}}}{h_{,}^{\frac{1}{2}}}\left(1 - \frac{1}{2}\,\frac{d\phi}{d\tau} + \frac{3}{8}\,\frac{d\phi^2}{d\tau^2}\right)$.

Also, $\frac{d(\rho^2)}{d\tau} = \frac{d(\rho_{,}^2\xi^2)}{d\tau} = \xi^2\,\frac{d(\rho_{,}^2)}{d\tau} + \rho_{,}^2\frac{d(\xi^2)}{d\tau} = \xi^2\,\frac{d(\rho_{,}^2)}{d\zeta}\,\frac{d\zeta}{d\tau} + \rho_{,}^2\,\frac{d(\xi^2)}{d\tau} = \frac{h}{h_{,}}\,\frac{d(\rho_{,}^2)}{d\zeta} + \rho_{,}^2\frac{d(\xi^2)}{d\tau}$. These values are to be substituted in that of S, after we have changed ρ into $\rho_{,}\xi$, v into $v_{,} + \epsilon nt$, and λ into $\lambda_{,} + \epsilon nt$. Make, therefore,

$$(S) = \left\{\frac{h_{,}^{\frac{1}{2}}}{h^{\frac{1}{2}}} + \frac{2\mu\rho_{,}}{h^2} - \left(\frac{2}{r} + \frac{2\mu}{h^2}\right)\rho_{,}\cos(\lambda_{,} - v_{,})\right\}\frac{1}{h_{,}^{\frac{1}{2}}h^{\frac{1}{2}}}\,\frac{dR}{dv} + \sin(\lambda_{,} - v_{,})\,\frac{2\rho_{,}}{h_{,}^{\frac{1}{2}}h^{\frac{3}{2}}}\,\frac{dR}{dr} - \frac{\epsilon n}{h_{,}}\,\frac{d(\rho_{,}^2)}{d\tau},\ \text{and}$$

$$(T)=\left\{\frac{2\mu\rho_{,}}{h^{2}}-\left(\frac{2}{r}+\frac{2\mu}{h^{2}}\right)\rho_{,}\cos\left(\lambda_{,}-v_{,}\right)\right\}\frac{1}{h_{,}^{\frac{1}{2}}h^{\frac{1}{2}}}\frac{dR}{dv}+\sin\left(\lambda_{,}-v_{,}\right)\frac{2\rho_{,}}{h_{,}^{\frac{1}{2}}h^{\frac{1}{2}}}\frac{dR}{dr}=$$
$$(S)-\frac{1}{h}\frac{dR}{dv}+\frac{\epsilon n}{h_{,}}\frac{d(\rho_{,}^{2})}{d\zeta}.$$

Putting the above values of ξ and $\frac{d(\rho^{2})}{d\tau}$ in that of S, after developing relative to ξ; we have, making $\rho_{,}^{0}$ the value of $\rho_{,}$ (a function of $\tau+\phi$) when ϕ is made nothing.

$$S=(S)-(T)\left(\tfrac{1}{2}\frac{d\phi}{d\tau}-\tfrac{3}{8}\frac{d\phi^{2}}{d\tau^{2}}\right)-\frac{\epsilon n}{h}\rho_{,}^{2}\frac{d(\xi^{2})}{d\tau}=$$
$$(S)-(T)\left(\tfrac{1}{2}\frac{d\phi}{d\tau}-\tfrac{3}{8}\frac{d\phi^{2}}{d\tau^{2}}\right)+\frac{\epsilon n\rho_{,}^{2}}{h_{,}}\frac{d^{2}\phi}{d\tau^{2}}\left(1-2\frac{d\phi}{d\tau}\right)=$$
$$(S)-(T)\left(\tfrac{1}{2}\frac{d\phi}{d\tau}-\tfrac{3}{8}\frac{d\phi^{2}}{d\tau^{2}}\right)+\frac{\epsilon n}{h_{,}}\rho_{,}^{2}\frac{d^{2}\phi}{d\tau^{2}}\left(1-2\frac{d\phi}{d\tau}\right)+\frac{\epsilon n\phi}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d^{2}\phi}{d\tau^{2}}.\quad(1)$$

We have now got rid of ξ. Let (S^{0}) and (T^{0}) be the values of (S) and (T) when ϕ is made nothing, or $\zeta=\tau$; then, by Taylor's theorem,

$$(S)=(S^{0})+\phi\frac{d(S^{0})}{d\tau}+\frac{\phi^{2}}{2}\frac{d^{2}(S^{0})}{d\tau^{2}},\quad(2)$$
$$(T)=(T^{0})+\phi\frac{d(T^{0})}{d\tau},$$

neglecting higher powers of the disturbing force; since we exclude those of the fourth order. Now ϕ enters nowhere but where it appears.

And putting $\frac{dh}{dt}$ for $-\frac{dR}{dv}$ for convenience,

$$(T^{0})=(S^{0})+\frac{1}{h}\frac{dh}{dt}+\frac{\epsilon n}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau},$$
$$\frac{d(T^{0})}{d\tau}=\frac{d(S^{0})}{d\tau}+\frac{\epsilon n}{h_{,}}\frac{d^{2}(\rho_{,}^{02})}{d\tau^{2}}$$

Therefore,

$$(T)=(S^{0})+\frac{1}{h}\frac{dh}{dt}+\phi\frac{d(S^{0})}{d\tau}+\frac{\epsilon n}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}+\frac{\epsilon n\phi}{h_{,}}\frac{d^{2}(\rho_{,}^{02})}{d\tau^{2}}.\quad(3)$$

Put in (1) the value of (S) given by (2), and that of (T) given by (3); and we have

$$S = (S^0)\left(1 - \tfrac{1}{2}\frac{d\phi}{d\tau} + \tfrac{3}{8}\frac{d\phi^2}{d\tau^2}\right) + \phi\frac{d(S^0)}{d\tau}\left(1 - \tfrac{1}{2}\frac{d\phi}{d\tau}\right) - \frac{1}{2h}\frac{dh}{dt}\frac{d\phi}{d\tau}\left(1 - \tfrac{3}{4}\frac{d\phi}{d\tau}\right) -$$
$$\frac{\mathfrak{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d\phi}{d\tau}\left(1 - \tfrac{3}{4}\frac{d\phi}{d\tau}\right) + \frac{\mathfrak{E}n\rho_{,}^{02}}{h_{,}}\frac{d^2\phi}{d\tau^2}\left(1 - 2\frac{d\phi}{d\tau}\right) + \frac{\mathfrak{E}n\phi}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d^2\phi}{d\tau^2} +$$
$$\frac{\phi^2}{2}\frac{d^2(S^0)}{d\tau^2} - \frac{\mathfrak{E}n\phi}{2h_{,}}\frac{d^2(\rho_{,}^{02})}{d\tau^2}\frac{d\phi}{d\tau};$$

and

$$S\frac{d\phi}{d\tau} = (S^0)\left(\frac{d\phi}{d\tau} - \tfrac{1}{2}\frac{d\phi^2}{d\tau^2}\right) + \phi\frac{d(S^0)}{d\tau}\frac{d\phi}{d\tau} - \frac{1}{2h}\frac{dh}{dt}\frac{d\phi^2}{d\tau^2} -$$
$$\frac{\mathfrak{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d\phi^2}{d\tau^2} + \frac{\mathfrak{E}n\rho_{,}^{02}}{h_{,}}\frac{d\phi}{d\tau}\frac{d^2\phi}{d\tau^2}.$$

We must now put these values of S and $S\frac{d\phi}{d\tau}$ in (11) of the last section, and we shall find as the result,

$$\frac{d^2\phi}{d\tau dt} = (S^0)\left(1 + \tfrac{1}{2}\frac{d\phi}{d\tau} - \tfrac{1}{8}\frac{d\phi^2}{d\tau^2}\right) + \phi\frac{d(S^0)}{d\tau}\left(1 + \tfrac{1}{2}\frac{d\phi}{d\tau}\right) - \frac{1}{2h}\frac{dh}{dt}\frac{d\phi}{d\tau}\left(1 + \tfrac{1}{4}\frac{d\phi}{d\tau}\right) +$$
$$\frac{d\phi}{dt}\frac{d^2\phi}{d\tau^2}\left(1 - \frac{d\phi}{d\tau}\right) - \frac{\mathfrak{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d\phi}{d\tau}\left(1 + \tfrac{1}{4}\frac{d\phi}{d\tau}\right) + \frac{\mathfrak{E}n\rho_{,}^{02}}{h_{,}}\frac{d^2\phi}{d\tau^2}\left(1 - \frac{d\phi}{d\tau}\right) + \frac{\phi^2}{2}\frac{d^2(S^0)}{d\tau^2} -$$
$$\frac{\mathfrak{E}n}{h_{,}}\frac{d^2(\rho_{,}^{02})}{d\tau^2}\frac{\phi}{2}\frac{d\phi}{d\tau} + \frac{\mathfrak{E}n\phi}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d^2\phi}{d\tau^2}. \qquad (4)$$

We can do nothing with the last in its present form, on account of the terms which contain ϕ and $\frac{d\phi}{dt}$. But happily these may be driven out by a little transformation. Neglecting terms containing the third power of the disturbing force, (4) gives

$$(S^0) = \frac{d^2\phi}{d\tau dt} - \tfrac{1}{2}(S^0)\frac{d\phi}{d\tau} - \phi\frac{d(S^0)}{d\tau} - \frac{d\phi}{d\tau}\frac{d^2\phi}{d\tau^2} + \frac{1}{2h}\frac{dh}{dt}\frac{d\phi}{dt} - \frac{\mathfrak{E}n\rho_{,}^{02}}{h_{,}}\frac{d^2\phi}{d\tau^2}$$
$$+ \frac{\mathfrak{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d\phi}{d\tau} =$$
$$\frac{d^2\phi}{d\tau dt} - \tfrac{1}{2}\frac{d\phi}{d\tau}\frac{d^2\phi}{d\tau dt} - \phi\frac{d^3\phi}{d\tau^2 dt} - \frac{d\phi}{dt}\frac{d^2\phi}{d\tau^2} + \frac{1}{2h}\frac{dh}{dt}\frac{d\phi}{d\tau} - \frac{\mathfrak{E}n\rho_{,}^{02}}{h_{,}}\frac{d^2\phi}{d\tau^2} + \frac{\mathfrak{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d\phi}{d\tau}.$$

With this value we eliminate (S^0) from (4), in those terms where the disturbing force rises above the first power, and we thus find

$$\frac{d^2\phi}{d\tau dt} = (S^0) + \frac{d}{dt}\left\{\tfrac{1}{4}\frac{d\phi^2}{d\tau^2} + \phi\frac{d^2\phi}{d\tau^2} - \tfrac{1}{8}\frac{d\phi^3}{d\tau^3} - \tfrac{3}{2}\phi\frac{d\phi}{d\tau}\frac{d^2\phi}{d\tau^2} - \tfrac{1}{2}\phi^2\frac{d^3\phi}{d\tau^3}\right\} -$$
$$\frac{1}{2h}\frac{dh}{dt}\left\{\frac{d\phi}{d\tau}\left(1 - \tfrac{1}{4}\frac{d\phi}{d\tau}\right) - \phi\frac{d^2\phi}{d\tau^2}\right\} + \frac{\mathcal{E}n\rho_{,}^{02}}{h_{,}}\left\{\frac{d^2\phi}{d\tau^2}\left(1 - \tfrac{3}{2}\frac{d\phi}{d\tau}\right) - \phi\frac{d^3\phi}{d\tau^3}\right\} -$$
$$\frac{\mathcal{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\left\{\frac{d\phi}{d\tau}\left(1 - \tfrac{1}{4}\frac{d\phi}{d\tau}\right) - \phi\frac{d^2\phi}{d\tau^2}\right\}. \qquad (5)$$

This has been put under the simplest form. It still contains three terms which we cannot manage; but these may be easily eliminated. Neglecting the powers of the disturbing force above the first, and integrating relative to t, adding the correction $\frac{d\phi_0}{d\tau}$, or the value of $\frac{d\phi}{d\tau}$ when the disturbing force is made to vanish, we have

$$\frac{d\phi}{d\tau} = \frac{d\phi_0}{d\tau} + \int (S^0)\,dt = W \text{ suppose.} \qquad (6)$$

Now if we include both the first and second powers of the disturbing force, we have

$$\frac{d\phi}{d\tau} = W + \tfrac{1}{4}\frac{d\phi^2}{d\tau^2} + \phi\frac{d^2\phi}{d\tau^2} - \tfrac{1}{2}\int\frac{d\phi}{d\tau}\frac{dh}{h} + \frac{\mathcal{E}n\rho_{,}^{02}}{h_{,}}\int\frac{d^2\phi}{d\tau^2}dt - \frac{\mathcal{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\int\frac{d\phi}{d\tau}dt,$$

which gives us immediately,

$$\frac{d\phi}{d\tau} - \tfrac{1}{4}\frac{d\phi^2}{d\tau^2} - \phi\frac{d^2\phi}{d\tau^2} = W - \tfrac{1}{2}\int\frac{d\phi}{d\tau}\frac{dh}{h} + \frac{\mathcal{E}n\rho_{,}^{02}}{h_{,}}\int\frac{d^2\phi}{d\tau^2}dt - \frac{\mathcal{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\int\frac{d\phi}{d\tau}dt.$$

And

$$\frac{d^2\phi}{d\tau^2} - \tfrac{3}{2}\frac{d\phi}{d\tau}\frac{d^2\phi}{d\tau^2} - \phi\frac{d^3\phi}{d\tau^3} = \frac{d}{d\tau}\left\{\frac{d\phi}{d\tau} - \tfrac{1}{4}\frac{d\phi^2}{d\tau^2} - \phi\frac{d^2\phi}{d\tau^2}\right\}.$$

Make

$$\frac{d\phi}{d\tau} - \tfrac{1}{4}\frac{d\phi^2}{d\tau^2} - \phi\frac{d^2\phi}{d\tau^2} = X.$$

Now as all the terms in the value of X after W are of the second order, and will be multiplied in (5) by quantities of the first order, we may put W for $\frac{d\phi}{d\tau}$ in the value of this quantity. Then

$$X = W - \tfrac{1}{2}\int W\frac{dh}{h} + \frac{\mathcal{E}n\rho_{,}^{02}}{h_{,}}\int\frac{dW}{d\tau}dt - \frac{\mathcal{E}n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\int W dt, \qquad (7)$$

and (5) will become

$$\frac{d^2\phi}{d\tau dt}=(S^0)+\frac{d}{dt}\left(\tfrac{1}{4}\frac{d\phi^2}{d\tau^2}+\phi\frac{d^2\phi}{d\tau^2}-\tfrac{1}{8}\frac{d\phi^3}{d\tau^3}-\tfrac{5}{2}\phi\frac{d\phi}{d\tau}\frac{d^2\phi}{d\tau^2}-\tfrac{1}{2}\phi^2\frac{d^3\phi}{d\tau^3}\right)$$
$$-\frac{1}{2h}X\frac{dh}{dt}-\frac{\epsilon n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}X+\frac{\epsilon n\rho_{,}^{02}}{h_{,}}\frac{dX}{d\tau}. \qquad (8)$$

We must now eliminate $\frac{d\phi^2}{d\tau^2}$, $\frac{d^2\phi}{d\tau^2}$, &c. For this purpose we take

$$\frac{d\phi}{d\tau}=X+\tfrac{1}{4}\frac{d\phi^2}{d\tau^2}+\phi\frac{d^2\phi}{d\tau^2},$$

which squared, neglecting terms above the third order, gives

$$\frac{d\phi^2}{d\tau^2}=X^2+\tfrac{1}{2}X\frac{d\phi^2}{d\tau^2}+2\phi X\frac{d^2\phi}{d\tau^2}=$$
$$X^2+\tfrac{1}{2}W\frac{d\phi^2}{d\tau^2}+2\phi W\frac{d^2\phi}{d\tau^2}=$$
$$X^2+\tfrac{1}{2}W^3+2\phi W\frac{dW}{d\tau}=$$
$$-W^2+2WX+\tfrac{1}{2}W^3+2\phi W\frac{dW}{d\tau}.$$
$$\frac{d^2\phi}{d\tau^2}=\frac{dX}{d\tau}+\tfrac{3}{2}\frac{d\phi}{d\tau}\frac{d^2\phi}{d\tau^2}+\phi\frac{d^3\phi}{d\tau^3}=$$
$$\frac{dX}{d\tau}+\tfrac{3}{2}W\frac{dW}{d\tau}+\phi\frac{d^2W}{d\tau^2}.$$

Substituting these values in (8), it becomes

$$\frac{d^2\phi}{d\tau dt}=(S^0)+\frac{d}{dt}\left(-\tfrac{1}{4}W^2+\tfrac{1}{2}WX+\phi\frac{dX}{d\tau}+\tfrac{1}{2}\phi W\frac{dW}{d\tau}+\tfrac{1}{2}\phi^2\frac{d^2W}{d\tau^2}\right)-$$
$$\frac{1}{2h}X\frac{dh}{dt}-\frac{\epsilon n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}X+\frac{\epsilon n\rho_{,}^{02}}{h_{,}}\frac{dX}{d\tau}.$$

Let

$$Y=-\tfrac{1}{4}W^2+\tfrac{1}{2}WX-\tfrac{1}{2}\int X\frac{dh}{h}-\frac{\epsilon n}{2h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\int Xdt+\frac{\epsilon n\rho_{,}^{02}}{h_{,}}\int\frac{dX}{d\tau}dt. \qquad (9)$$

By means of this the integral of the preceding is

$$\frac{d\phi}{d\tau}=W+\phi\left(\frac{dX}{d\tau}+\tfrac{1}{2}W\frac{dW}{d\tau}\right)+\tfrac{1}{2}\phi^2\frac{d^2W}{d\tau^2}+Y. \qquad (10)$$

Since $\overline{\left(\frac{d\phi}{d\tau}\right)} = \frac{dw}{dt} + \frac{\mathcal{E}nr_{,}^{2}}{h_{,}}$; changing τ into t, we have

$$\frac{dw}{dt} = \overline{(W)} - \frac{\mathcal{E}nr_{,}^{2}}{h_{,}} + w\,\overline{\left(\frac{dX}{d\tau} + \tfrac{1}{2}W\frac{dW}{d\tau}\right)} + \tfrac{1}{2}w^{2}\,\overline{\left(\frac{d^{2}W}{d\tau^{2}}\right)} + \overline{(Y)}. \tag{11}$$

We have now got entirely quit of ξ and ϕ; and τ is contained only in $\rho_{,}^{0}$ and $\lambda_{,}^{0}$, which are the same functions of τ and $a_{,}$, $e_{,}$, &c., as in an undisturbed orbit; and we may everywhere put them without the sign of integration, and then change τ into t. Thus we have virtually got rid of τ, and the result may be put under the same form as if it were obtained with only the ordinary time t.

If we change z into $t + w$, (3) of section (1) gives

$$\frac{h}{h_{,}\beta^{2}} = 1 + \frac{dw}{dt} + \frac{\mathcal{E}nr_{,}^{2}}{h_{,}},$$

and

$$\beta = \frac{h^{\frac{1}{2}}}{h_{,}^{\frac{1}{2}}}\left\{1 - \tfrac{1}{2}\left(\frac{dw}{dt} + \frac{\mathcal{E}nr_{,}^{2}}{h_{,}}\right) + \text{\&c.}\right\};$$

which will give β when $\frac{dw}{dt}$ is known. And we may change β into ϵ^{β}, and take the logarithm of both members, if we wish to have a result in the experimental form, which is that of M. Hansen. Or we may change β into $1 + \frac{\beta}{r_{,}}$; then we have

$$r = r_{,} + \beta.$$

But, changing ζ into $\tau + \phi$, we may employ (2) of section (1), which gives

$$\frac{h}{h_{,}\xi^{2}} = 1 + \frac{d\phi}{d\tau},$$

and

$$\xi = \frac{h^{\frac{1}{2}}}{h_{,}^{\frac{1}{2}}}\left(1 + \frac{d\phi}{d\tau}\right)^{-\frac{1}{2}}.$$

If we wish to have a result in the experimental form, this is, perhaps, the most convenient. And thus changing ξ into ϵ^{ξ}, and taking the logarithm of both members, we have

$$\xi = \tfrac{1}{2}\log\left(\frac{h}{h_{,,}}\right) - \tfrac{1}{2}\log\left(1 + \frac{d\phi}{d\tau}\right).$$

M. Hansen employs this method, and differentiates relative to τ, by which means he gets rid of $\frac{1}{2}\log\left(\frac{h}{h_{,}}\right)$. Thus we have

$$\frac{d\xi}{d\tau}=-\frac{1}{2}\,\frac{\frac{d^2\phi}{d\tau^2}}{1+\frac{d\phi}{d\tau}},$$

and changing τ into t,

$$\overline{\left(\frac{d\xi}{d\tau}\right)}=-\frac{1}{2}\,\overline{\left(\frac{\frac{d^2\phi}{d\tau^2}}{1+\frac{d\phi}{d\tau}}\right)}.$$

Or, by (7) of the section referred to,

$$\frac{d\beta}{dt}=\frac{\mathcal{E}nr}{h_{,}}\,\frac{dr_{,}}{dz}-\frac{1}{2}\,\overline{\left(\frac{\frac{d^2\phi}{d\tau^2}}{1+\frac{d\phi}{d\tau}}\right)}.$$

In this form M. Hansen leaves it. But we might employ (8) of the same section to reduce it, or to give a distinct form; but that would introduce $\frac{h}{h_{,}}$ again.

4. We now proceed to the determination of the latitude, and the reduction to a fixed plane. Make σ the sine of the latitude, i the inclination, and ϑ and θ the longitude of the node on the plane of the orbit and on the fixed plane respectively, ϑ having an origin fixed on the former plane. By making the plane of the orbit to turn round the radius vector an infinitesimal space, it is easily seen that,

$$d\vartheta=\cos i\,d\theta\,; \tag{1}$$

and we have the known formulæ,

$$\left.\begin{aligned}\frac{di}{dt}&=\frac{1}{h\sin i}\,\frac{dR}{d\theta}=\frac{\cos i}{h\sin i}\,\frac{dR}{d\vartheta},\\ \frac{d\vartheta}{dt}&=-\frac{\cos i}{h\sin i}\,\frac{dR}{di}.\end{aligned}\right\} \tag{2}$$

I shall give some theorems here similar to (6) of the first section, without, however, making any use of them.

We have $\sigma = \sin i \sin (v - \vartheta) = \sin i \, (\sin v \cos \vartheta - \cos v \sin \vartheta)$, $\frac{d\sigma}{dt} = \sin i \cos (v - \vartheta) \frac{dv}{dt} = \frac{h}{r^2} \sin i \cos (v - \vartheta)$. Therefore, $\frac{r^2}{h} \frac{d\sigma}{dt} = \sin i \cos (v - \vartheta) = \sin i \, (\cos v \cos \vartheta + \sin v \sin \vartheta)$. From these we easily find

$$\sigma \sin v + \frac{r^2}{h} \frac{d\sigma}{dt} \cos v = \sin i \cos \vartheta.$$

$$- \sigma \cos v + \frac{r^2}{h} \frac{d\sigma}{dt} \sin v = \sin i \sin \vartheta.$$

Again, let κ be the same function of τ and the variable elements that σ is of t and the same elements ; so that

$$\kappa = \sin i \sin (\lambda - \vartheta), \ \frac{d\kappa}{d\tau} = \frac{h}{\rho^2} \sin i \cos (\lambda - \vartheta);$$

and, as before, we shall have

$$\kappa \sin \lambda + \frac{\rho^2}{h} \frac{d\kappa}{d\tau} \cos \lambda = \sin i \cos \vartheta,$$

$$- \kappa \cos \lambda + \frac{\rho^2}{h} \frac{d\kappa}{d\tau} \sin \lambda = \sin i \sin \vartheta.$$

Equating the two sets of values of $\sin i \cos \vartheta$ and $\sin i \sin \vartheta$,

$$\sigma \sin v + \frac{r^2}{h} \frac{d\sigma}{dt} \cos v = \kappa \sin \lambda + \frac{\rho^2}{h} \frac{d\kappa}{d\tau} \cos \lambda,$$

$$\sigma \cos v - \frac{r^2}{h} \frac{d\sigma}{dt} \sin v = \kappa \cos \lambda - \frac{\rho^2}{h} \frac{d\kappa}{d\tau} \sin \lambda.$$

By multiplying these by $\sin v$, $\cos v$, $\sin \lambda$, $\cos \lambda$, and adding and subtracting the products, we shall have no difficulty in deducing

$$\sigma = \kappa \cos (\lambda - v) \frac{\rho^2}{h} - \frac{d\kappa}{d\tau} \sin (\lambda - v).$$

$$\frac{r^2}{h} \frac{d\sigma}{dt} = \kappa \sin (\lambda - v) + \frac{\rho^2}{h} \frac{d\kappa}{d\tau} \cos (\lambda - v).$$

$$\kappa = \sigma \cos (\lambda - v) + \frac{r^2}{h}\frac{d\sigma}{dt} \sin (\lambda - v),$$

$$\frac{\rho^2}{h}\frac{d\kappa}{d\tau} = - \sigma \sin (\lambda - v) + \frac{r^2}{h}\frac{d\sigma}{dt} \cos (\lambda - v).$$

These two sets of equations correspond to (5) and (6) of the first section.

From $\frac{r^2}{h}\frac{d\sigma}{dt} = \sin i \cos (v - \vartheta)$, $\frac{d\sigma}{d\vartheta} = - \sin i \cos (v - \vartheta)$, and $\frac{d\sigma}{di} = \cos i$ $\sin (v - \vartheta)$; we easily find

$$\left.\begin{array}{l} \frac{d\sigma}{d\vartheta} = - \frac{r^2}{h}\frac{d\sigma}{dt}, \quad \frac{d\sigma}{di} = \sigma \frac{\cos i}{\sin i}, \\ \frac{d\kappa}{d\vartheta} = - \frac{\rho^2}{h}\frac{d\kappa}{d\tau}, \quad \frac{d\kappa}{di} = \kappa \frac{\cos i}{\sin i}, \end{array}\right\} \quad (3)$$

which may be found useful.

The part of the disturbance function depending on the inclination of the orbit to the fixed plane is very troublesome to express by means of i, ϑ, and θ, and the corresponding quantities i', ϑ', and θ', relative to the disturbing body; and when either ϑ is expressed by means of θ, ϑ' by means of θ', or the latter by means of the former, would contain a great number of terms. But it may be expressed very simply without these quantities, by means of the latitude ϕ and the reduction Δ, and the corresponding quantities ϕ' and Δ' relative to the disturbing body. Thus we should have $v - \Delta$, and $v' - \Delta'$, and $v' - \Delta'$ for the longitudes on the fixed plane, and

$$\frac{x}{r} = \cos \phi \cos (v - \Delta), \quad \frac{y}{r} = \cos \phi \sin (v - \Delta), \quad \frac{z}{r} = \sin \phi,$$

$$\frac{x'}{r'} = \cos \phi' \cos (v' - \Delta'), \quad \frac{y'}{r'} = \cos \phi' \sin (v' - \Delta'), \quad \frac{z'}{r'} = \sin \phi'.$$

But $\cos \chi = \frac{xx' + yy' + zz'}{rr'}$; therefore,

$$\cos \chi = \cos \phi \cos \phi' \{\cos (v - \Delta) \cos (v' - \Delta') + \sin (v - \Delta) \sin (v' - \Delta')\} +$$
$$\sin \phi \sin \phi' = \cos \phi \cos \phi' \cos (v - v' - \Delta + \Delta') + \sin \phi \sin \phi' = (1 - \sigma^2)^{\frac{1}{2}}$$
$$(1 - \sigma'^2)^{\frac{1}{2}} \left\{ \cos (v - v') + \sin (v - v') (\Delta - \Delta') - \cos (v - v') \frac{(\Delta - \Delta')^2}{2} - \&c. \right\} + \sigma\sigma'. \quad (4)$$

We suppose here that σ and σ', and therefore also Δ and Δ', are small, as in the case of the moon and the old planets.

Making for a moment $r_{,}$ the projection of the radius vector, and $v_{,}$ the longitude on the fixed plane, and comparing the differentials of the areas described on this plane and that of the orbit, we have

$$\cos i \, r^2 dv = r_{,}^2 \, dv_{,} = r^2 \cos^2 \phi \, dv_{,},$$

and $\cos i \, dv = \cos^2 \phi \, dv_{,} = \cos^2 \phi \, (dv - d\Delta)$; or

$$(\cos^2\phi - \cos i)\frac{dv}{dt} = \cos^2\phi \frac{d\Delta}{dt}; \text{ or } \left(2\sin^2\frac{i}{2} - \sin^2\phi\right)\frac{h}{r^2} = \cos^2\phi\frac{d\Delta}{dt}.$$

Whence we derive

$$2\sin^2\frac{i}{2} = \sin^2\phi + \frac{r^2}{h}\,\frac{d\Delta}{dt}\cos^2\phi,$$

$$\frac{d\Delta}{dt} = \frac{h}{r^2\cos^2\phi}\left(2\sin^2\frac{i}{2} - \sin^2\phi\right).$$

And putting σ^2 for $\sin^2\phi$, $1-\sigma^2$ for $\cos^2\phi$,

$$\left.\begin{aligned} 2\sin^2\frac{i}{2} &= \sigma^2 + \frac{r^2}{h}\,\frac{d\Delta}{dt}(1-\sigma^2), \\ \frac{d\Delta}{dt} &= \frac{h}{r^2(1-\sigma^2)}\left(2\sin^2\frac{i}{2} - \sigma^2\right). \end{aligned}\right\} \qquad (5)$$

The equation $d\vartheta = \cos i d\theta$ gives $d\theta = \frac{1}{\cos i}d\vartheta$, and $d(\theta-\vartheta) = \left(\frac{1}{\cos i} - 1\right)d\vartheta$; or, $d(\theta - \vartheta) = \frac{2\sin^2\frac{t}{2}}{\cos i}d\vartheta$. And by integration

$$\theta - \vartheta = 2\int\frac{\sin^2\frac{i}{2}}{\cos i}d\vartheta = \alpha \text{ suppose.} \qquad (6)$$

Let $\Delta_{,} = (v - \vartheta) - (v_{,} - \theta)$, where $v_{,} = v - \Delta$. Then

$$\sin\Delta_{,} = \sin(v-\vartheta)\cos(v_{,}-\theta) - \cos(v-\vartheta)\sin(v_{,}-\theta).$$

But by the well-known theorems of spherical trigonometry, $\sin(v - \vartheta) =$

$\frac{\sin\phi}{\sin i}$, $\sin(v_{,} - \theta) = \frac{\tan\phi}{\tan i}$, and $\cos(v_{,} - \theta) = \frac{\cos(x - \vartheta)}{\cos\phi}$. Therefore, by substitution,

$$\sin\Delta_{,} = \tan\phi\cos(v - \vartheta)\left(\frac{1}{\sin i} - \frac{1}{\tan i}\right) = \tan\frac{i}{2}\tan\phi\cos(v - \vartheta) =$$

$$\frac{\sin i}{2\cos^2\frac{i}{2}}\tan\phi\cos(v - \vartheta).$$

But $\sin\Delta_{,} = \Delta_{,} - \frac{1}{6}\Delta_{,}^3$, and $\sin i\cos(v - \vartheta) = \frac{r^2}{h}\frac{d\sigma}{dt}$. Consequently, ${}_{,}\Delta -$

$$\frac{1}{6}\Delta_{,}^3 = \frac{\tan\phi}{2\cos^2\frac{i}{2}}\frac{r^2}{h}\frac{d\sigma}{dt} = \frac{r^2}{2h\cos^2\frac{i}{2}}\frac{\sigma}{\sqrt{(1-\sigma^2)}}\frac{d\sigma}{dt} = -\frac{r^2}{2h\cos^2\frac{i}{2}}\frac{d\sqrt{(1-\sigma^2)}}{dt},$$

and $\Delta_{,} = -\frac{r^2}{2h\cos^2\frac{i}{2}}\frac{d\sqrt{(1-\sigma^2)}}{dt} + \frac{1}{6}\Delta_{,}^3$.

Now $\Delta_{,} = v - v_{,} - \vartheta + \theta = \Delta - \vartheta + \theta = \Delta + \alpha$; and, therefore,

$$\Delta = -\frac{r^2}{2h\cos^2\frac{i}{2}}\frac{d\sqrt{(1-\sigma^2)}}{dt} - \alpha + \frac{1}{6}\Delta^3; \qquad (7)$$

neglecting smaller quantities, observing that α is of the order of the disturbing force multiplied by $\sin^2\frac{i}{2}$.

We want now to find $\frac{d\Delta}{di}$ and $\frac{d\Delta}{d\vartheta}$. To do this, we shall transpose $\frac{1}{6}\Delta^3$, and, after taking the partial differential, divide by $1 - \frac{1}{2}\Delta^2$, or multiply by $1 + \frac{1}{2}\Delta^2$; and by means of $\frac{d^2\sigma}{didt} = \frac{h}{r^2}\cos i\cos(v - \vartheta) = \frac{\cos i}{\sin i}\frac{d\sigma}{dt}$, and $\frac{d^2\sigma}{d\vartheta dt} = \frac{h}{r^2}\sigma$, we shall find

$$\frac{d\Delta}{di} = \frac{r^2}{2h\cos^2\frac{i}{2}}\frac{d\sigma}{dt}\frac{\sigma}{\sqrt{(1-\sigma^2)}}\left\{\cot i\left(\frac{2-\sigma^2}{1-\sigma^2}\right) + \tan\frac{i}{2}\right\}(1 + \tfrac{1}{2}\Delta^2). \qquad (8)$$

$$\frac{d\Delta}{d\vartheta}=\frac{1}{2\cos^2\frac{i}{2}}\cdot\frac{1}{\sqrt{(1-\sigma^2)}}\left(\sigma^2-\frac{r^4}{h^2}\frac{d\sigma^2}{dt^2}\frac{1}{1-\sigma^2}\right)(1+\tfrac{1}{2}\Delta^2)-\frac{2\sin^2\frac{i}{2}}{\cos i}(1+\tfrac{1}{2}\Delta^2).$$

But $\frac{r^4}{h^2}\frac{d\sigma^2}{dt^2}=\sin^2 i\cos^2(v-\vartheta)=\sin^2 i-\sin^2 i\sin^2(v-\vartheta)=\sin^2 i-\sigma^2.$ Therefore, putting this value in the preceding, we have

$$\frac{d\Delta}{d\vartheta}=\frac{1}{2\cos^2\frac{i}{2}}\cdot\frac{1}{\sqrt{(1-\sigma^2)}}\left(\frac{\sigma^2-\sin^2 i}{1-\sigma^2}+\sigma^2\right)(1+\tfrac{1}{2}\Delta^2)-\frac{2\sin^2\frac{i}{2}}{\cos i}(1+\tfrac{1}{2}\Delta^2). \quad (9)$$

But the formulæ we have obtained are not convenient for actual application; it may be well, therefore, to give them in series. For this purpose we make $\sin i=s$, $\sin(v-\vartheta)=\eta$; then $\sigma=s\eta$. But as, in the first differentials, the elements do not vary, we must make $\frac{d\sigma}{dt}=s\frac{d\eta}{dt}$, and in $\frac{d\eta}{dt}$ we must not differentiate ϑ.

Now, expanding $\sqrt{(1-\sigma^2)}$ in (7), we find

$$\Delta=\frac{r^2}{2h\cos^2\frac{i}{2}}\cdot\frac{d\sigma}{dt}(\sigma+\tfrac{1}{2}\sigma^3+\tfrac{3}{8}\sigma^5)-a+\tfrac{1}{6}\Delta^3,$$

neglecting smaller terms. Or,

$$\Delta=\frac{r^2s^2}{2h\cos^2\frac{i}{2}}\cdot\frac{d\eta}{dt}(\eta+\tfrac{1}{2}s^2\eta^3+\tfrac{3}{8}s^4\eta^5)-a+\tfrac{1}{6}\Delta^3.$$

But

$$\frac{1}{2\cos^2\frac{i}{2}}=\frac{2\sin^2\frac{i}{2}}{4\sin^2\frac{i}{2}\cos^2\frac{i}{2}}=\frac{2\sin^2\frac{i}{2}}{s^2}=\frac{1-\cos i}{s^2}=\frac{1-\sqrt{(1-s^2)}}{s^2}=\tfrac{1}{2}+\tfrac{1}{8}s^2+\tfrac{1}{16}s^4.$$

Substituting this value, we have at length,

$$\Delta=\frac{r^2s^2}{h}\frac{d\eta}{dt}\{\tfrac{1}{2}\eta+s^2(\tfrac{1}{8}\eta+\tfrac{1}{4}\eta^3)+s^4(\tfrac{1}{16}\eta+\tfrac{1}{16}\eta^3+\tfrac{3}{16}\eta^5)\}-a+\tfrac{1}{6}\Delta^3; \quad (10)$$

where some small quantities are retained, which, perhaps, are not wanted. In taking the partial differential coefficients of Δ, it will be better to find $\frac{d\Delta}{ds}$, and not $\frac{d\Delta}{di}$.

$$\frac{d\Delta}{ds}=\frac{2r^2s}{h}\frac{d\eta}{dt}\{\tfrac{1}{2}\eta+s^2(\eta+\tfrac{1}{2}\eta^3)+s^4(\tfrac{3}{16}\eta+\tfrac{3}{16}\eta^3+\tfrac{9}{16}\eta^5)\}+\tfrac{1}{2}\Delta^2\frac{d\Delta}{ds}.$$

Write this for a moment, $\frac{d\Delta}{ds}=M+\frac{1}{2}\Delta^2\frac{d\Delta}{ds}$. Transpose the last term of the second member. Then $(1-\frac{1}{2}\Delta^2)\frac{d\Delta}{ds}=M$; whence $\frac{d\Delta}{ds}=M(1-\frac{1}{2}\Delta^2)^{-1}=M(1+\frac{1}{2}\Delta^2)$; and

$$\frac{d\Delta}{ds}=\frac{2rs}{h}\frac{d\eta}{dt}\{\tfrac{1}{2}\eta+s^2(\eta+\tfrac{1}{2}\eta^3)+s^4(\tfrac{3}{16}\eta+\tfrac{3}{16}\eta^3+\tfrac{9}{16}\eta^5)\}(1+\tfrac{1}{2}\Delta^2). \quad (11)$$

Preparatory to finding $\frac{d\Delta}{d\vartheta}$, we may observe that $\frac{d\eta}{dt}=\cos(v-\vartheta)\frac{dv}{dt}=\frac{h}{r^2}\cos(v-\vartheta)$; and, therefore, $\frac{d^2\eta}{d\vartheta dt}=\frac{h}{r^2}\sin(v-\vartheta)=\frac{h}{r^2}\eta$. Also, $\frac{d\eta}{d\vartheta}=-\cos(v-\vartheta)=-\frac{r^2}{h}\frac{d\eta}{dt}$; and $\frac{r^4}{h^2}\frac{d\eta^2}{dt^2}=\cos^2(v-\vartheta)=1-\sin^2(v-\vartheta)=1-\eta^2$. We now find

$$\frac{d\Delta}{d\vartheta}=s^2\eta\{\tfrac{1}{2}\eta+s^2(\tfrac{1}{2}\eta+\tfrac{1}{4}\eta^3)+s^4(\tfrac{1}{16}\eta+\tfrac{1}{16}\eta^3+\tfrac{3}{16}\eta^5)\}-\frac{2\sin^2\frac{i}{2}}{\cos i}+\tfrac{1}{2}\Delta^2\frac{d\Delta}{d\vartheta}-$$
$$s^2(1-\eta^2)\{\tfrac{1}{2}+s^2(\tfrac{1}{2}+\tfrac{3}{4}\eta^2)+s^4(\tfrac{1}{16}+\tfrac{3}{16}\eta^2+\tfrac{15}{16}\eta^4)\}.$$

By further reduction, and treating the equation with regard to Δ as in the last case,

$$\frac{d\Delta}{d\vartheta}=s^2\{-\tfrac{1}{2}+\eta^2+s^2(-\tfrac{1}{2}+\tfrac{1}{4}\eta^2+\eta^4)+s^4(-\tfrac{1}{16}-\tfrac{1}{16}\eta^2-\tfrac{11}{16}\eta^4+\tfrac{9}{8}\eta^6)\}$$
$$(1+\tfrac{1}{2}\Delta^2)-\frac{2\sin^2\frac{i}{2}}{\cos i}(1+\tfrac{1}{2}\Delta^2).$$

But $$\frac{2\sin^2\frac{i}{2}}{\cos i} = \frac{1-\cos i}{\cos i} = \frac{1}{\cos i} - 1 = \frac{1}{\sqrt{(1-s^2)}} - 1 = \tfrac{1}{2}s^2 + \tfrac{3}{8}s^4 - \tfrac{5}{16}s^6.$$
Therefore,

$$\frac{d\Delta}{d\vartheta} = s^2\{-1+\eta^2+s^2\left(-\tfrac{7}{8}+\tfrac{1}{4}\eta^2+\eta^4\right)+s^4\left(-\tfrac{3}{8}-\tfrac{1}{16}\eta^2-\tfrac{11}{16}\eta^4+\tfrac{9}{8}\eta^6\right)\}\left(1+\tfrac{1}{2}\Delta^2\right). \quad (12)$$

Since $2\sin^2\frac{i}{2} = 1-\cos i = 1-\sqrt{(1-s^2)} = \frac{1}{2}s^2+\frac{1}{8}s^4+\frac{1}{16}s^6$. Putting this value in (5), and developing the terms, it becomes

$$\frac{d\Delta}{dt} = \frac{h}{r^2}\{s^2\left(\tfrac{1}{2}-\eta^2\right)+s^4\left(\tfrac{1}{8}+\tfrac{1}{2}\eta^2-\eta^4\right)+s^6\left(\tfrac{1}{16}+\tfrac{1}{8}\eta^2+\tfrac{1}{2}\eta^4-\eta^6\right)\}. \quad (13)$$

We will now transform (2), so as to introduce σ and Δ, or rather η and Δ.

$$\frac{ds}{dt} = \cos i\,\frac{di}{dt} = \frac{\cos^2 i}{h\sin i}\,\frac{dR}{d\vartheta} = \frac{1-s^2}{hs}\,\frac{dR}{d\vartheta},$$
$$\frac{d\vartheta}{dt} = -\frac{\cos^2 i}{h\sin i}\,\frac{dR}{ds} = -\frac{1-s^2}{hs}\,\frac{dR}{ds}.$$

Or, putting $s\eta$ for σ in R, and introducing the partial differentials of the new quantities,

$$\left.\begin{aligned}\frac{ds}{dt} &= \frac{1-s^2}{hs}\left(\frac{dR}{d\eta}\,\frac{d\eta}{d\vartheta}+\frac{dR}{d\Delta}\,\frac{d\Delta}{d\vartheta}\right),\\ \frac{d\vartheta}{dt} &= -\frac{1-s^2}{hs}\left(\frac{dR}{ds}+\frac{dR}{d\Delta}\,\frac{d\Delta}{ds}\right).\end{aligned}\right\} \quad (14)$$

It may not be amiss to find $2\sin^2\frac{i}{2}$ separately. Thus, i_0 being the mean value of i, or its value when the disturbing force is nothing, and, therefore, a constant quantity; we have

$$2\sin^2\frac{i}{2} = 2\sin^2\frac{i_0}{2} + 2\int d\sin^2\frac{i}{2} = 2\sin^2\frac{i_0}{2} + \int \sin i\,di =$$

$$2\sin^2\frac{i_0}{2} + \int\frac{dt}{h}\cos i\,\frac{dR}{d\vartheta} = 2\sin^2\frac{i_0}{2} + \int\frac{dt}{h}\cos i\left(\frac{dR}{d\eta}\,\frac{d\eta}{d\vartheta}+\frac{dR}{d\Delta}\,\frac{d\Delta}{d\vartheta}\right).$$

Make $\frac{1}{h}\frac{dR}{d\eta}\frac{d\eta}{d\vartheta}+\frac{1}{h}\frac{dR}{d\Delta}\frac{d\Delta}{d\vartheta}=P$, to abridge. Then

$$2\sin^2\frac{i}{2}=2\sin^2\frac{i_0}{2}+\int\cos iPdt=2\sin^2\frac{i_0}{2}+\int Pdt-2\int\sin^2\frac{i}{2}Pdt.$$

But substituting this value of $2\sin^2\frac{i}{2}$ under the integral sign of the second member,

$$2\sin^2\frac{i}{2}=2\sin^2\frac{i_0}{2}+\cos i_0\int Pdt-\tfrac{1}{2}\left(\int Pdt\right)^2+2\int Pdt\int\sin^2\frac{i}{2}Pdt.$$

By continuing these operations, we find

$$2\sin^2\frac{i}{2}=2\sin^2\frac{i_0}{2}+\cos i_0\int Pdt-\tfrac{1}{2}\cos i_0\left(\int Pdt\right)^2+\tfrac{1}{2\cdot 3}\cos i_0\left(\int Pdt\right)^3-\&c. \quad (15)$$

This may serve to eliminate $\sin^2\frac{i}{2}$. But since $2\sin^2\frac{i}{2}=1-\cos i=1-\sqrt{(1-\sin^2 i)}=\frac{1}{2}\sin^2 i+\frac{1}{8}\sin^4 i+\frac{1}{16}\sin^6 i+\&c.$, it may also serve to eliminate $\sin^2 i$ and $\sin i$, if we should find it convenient to do so.

It is much easier to find r and v on the plane of the orbit, than to find the values of the corresponding quantities on the fixed plane; but it is more difficult to find the latitude in the former case. It is very troublesome to find it directly from the variable values of the elements i and ϑ; and yet we must have the values of these quantities separately to a considerable degree of exactness in order to find α. M. Hansen has found the latitude by means of $\sin i \sin\vartheta$, $\sin i\cos\vartheta$; which is a much better method; and he has found α by means of these latter quantities, or rather functions derived from them. But this is attended with a great deal of trouble. I propose to pursue a different course, and to find $\sin i$, or s, separately, by the equation given above for the purpose, and then to find η. To do which I shall find

$$y=\sin(x-\vartheta-\gamma nt),$$

x being a constant quantity. This is finding a function of $\vartheta+\gamma nt$.

$$\frac{dy}{dt}=\frac{dy}{d\vartheta}\left(\gamma n+\frac{d\vartheta}{dt}\right)=$$

$$\frac{dy}{d\vartheta}\left\{\gamma n-\frac{1-s^2}{hs}\left(\frac{dR}{ds}+\frac{dR}{d\Delta}\frac{d\Delta}{ds}\right)\right\}=\frac{dy}{d\vartheta}Q \text{ suppose.}$$

It must be observed, that γnt is the uniform regression of the node, and γ is to be so determined as to take away the constant terms from Q.

Now $y_0 = \sin(x - \vartheta_0)$ the value when the disturbing force is made nothing; therefore, integrating,

$$y = \sin(x - \vartheta_0) + \int \frac{dy}{d\vartheta} Q dt.$$

For a first approximation, we make $\frac{dy}{d\vartheta} = -\cos(x - \vartheta_0)$; and after the integration is performed, we may make x anything we please. We shall make $x = v + \gamma nt = v_{,} + \epsilon nt + \gamma nt$. Then $y = \sigma$, and we shall have

$$\sigma = \sin(v + \gamma nt - \vartheta_0) + \int \frac{dy}{d\vartheta} Q dt. \tag{16}$$

After this substitution has been made for x, and v replaced by $v_{,} + \epsilon nt$, $v_{,}$, being a given function of z, may be allowed to remain, or may be developed in terms of z, and z may be developed in terms of t.

We may make $y = \sin i \sin(x - \vartheta - \gamma nt) = \sin i \cos(\vartheta + \gamma nt) \sin x - \sin i \sin(\vartheta + \gamma nt) \cos x = p \sin x - q \cos x$.

$$p = \sin i \cos(\vartheta + \gamma nt),\ q = \sin i \sin(\vartheta + \gamma nt).$$

$$\frac{dp}{dt} = \cos i \cos(\vartheta + \gamma nt) \frac{di}{dt} - \sin i \sin(\vartheta + \gamma nt)\left(\frac{d\vartheta}{dt} + \gamma n\right).$$

$$\frac{dq}{dt} = \cos i \sin(\vartheta + \gamma nt) \frac{di}{dt} + \sin i \cos(\vartheta + \gamma nt)\left(\frac{d\vartheta}{dt} + \gamma n\right).$$

Or,

$$\frac{dp}{dt} = p \frac{\cos i}{\sin i} \frac{di}{dt} - q\left(\frac{d\vartheta}{dt} + \gamma n\right) = \frac{p}{s} \frac{ds}{dt} - q\left(\frac{d\vartheta}{dt} + \gamma n\right).$$

$$\frac{dq}{dt} = q \frac{\cos i}{\sin i} \frac{di}{dt} + p\left(\frac{d\vartheta}{dt} + \gamma n\right) = \frac{q}{s} \frac{ds}{dt} + p\left(\frac{d\vartheta}{dt} + \gamma n\right).$$

We may substitute in this, for $\frac{ds}{dt}$ and $\frac{d\vartheta}{dt}$, their values from (14); but as I prefer the former method, I shall not pursue this further; nor is it necessary, since any one who is desirous of doing it may easily carry it through. We may observe, however, that making δp, δq, δs, $\delta\vartheta$, the alterations produced in p, q, &c., by the disturbing force, we have for the first power of that force,

$$\delta p = \frac{p}{s}\,\delta s - q\delta\vartheta,\quad \delta q = \frac{q}{s}\,\delta s + p\delta\vartheta\,;$$

whence, observing that $s^2 = p^2 + q^2$, we have

$$\delta s = \frac{p\delta p + q\delta q}{s},\quad \delta\vartheta = \frac{p\delta q - q\delta p}{s^2}\,.$$

Thus we find the alterations produced in s and ϑ from those produced in p and q. And in like manner we may find the alterations depending on the second power of the disturbing force.

In this section we have taken no notice of the development of r in the formulæ where it has appeared. Making $r = r_{,}\,\beta$, we may either let it stand thus, or put from (3) section (1) its value in terms of $\frac{dw}{dt}$, which we easily find to be

$$\beta = \frac{h^{\frac{1}{2}}}{h_{,}^{\frac{1}{2}}}\left(1 + \frac{dw}{dt} + \frac{\epsilon n r_{,}}{h_{,}}\right)^{-\frac{1}{2}}.$$

Then we shall have only $r_{,}$ and $v_{,}$, besides the terms depending on the latitude, to develope. These are given functions of $z = t + w$, and may, therefore, be developed together by Taylor's theorem.

We might have developed the value of S in (11), section (2), relative to ξ and ϕ, differently.

Since the difference only of λ and v enters into the composition of S, we may change these quantities into $\lambda_{,}$ and $v_{,}$, or into the anomalies $\lambda_{,} - \pi_{,}$ and $v_{,} - \pi_{,}$. Using for simplicity $\lambda_{,}$ and $v_{,}$, and making

$$M = \left(\frac{1}{r} + \frac{\mu}{h^2}\right)\frac{2\cos v_{,}}{h}\,\frac{dR}{dv} + \frac{2\sin v_{,}}{h}\,\frac{dR}{dr},$$

$$N = \left(\frac{1}{r} + \frac{\mu}{h^2}\right)\frac{2\sin v_{,}}{h}\,\frac{dR}{dv} - \frac{2\cos v_{,}}{h}\,\frac{dR}{dr},$$

we have

$$S=\left(1+\frac{2\mu\rho}{h^2}\right)\frac{1}{h}\frac{dR}{dv}-\rho\cos\lambda_{,}M-\rho\sin\lambda_{,}N-\frac{\epsilon n}{h}\frac{d(\rho^2)}{d\tau}.$$

We eliminate ξ from this by putting $\rho_{,}\frac{h^{\frac{1}{2}}}{h_{,}^{\frac{1}{2}}}\left(1-\frac{1}{2}\frac{d\phi}{d\tau}+\frac{3}{8}\frac{d\phi^2}{d\tau^2}\right)$ for $\rho=\rho_{,}\xi$.

Let (S) be the value of S when ρ is changed into $\rho_{,}\frac{h^{\frac{1}{2}}}{h_{,}^{\frac{1}{2}}}$, and we have

$$S=(S)+\left\{(\rho_{,}\cos\lambda_{,}M+\rho_{,}\sin\lambda_{,}N)\frac{h^{\frac{1}{2}}}{2h_{,}^{\frac{1}{2}}}-\frac{\rho_{,}}{h_{,}^{\frac{1}{2}}h^{\frac{1}{2}}}\frac{\mu}{h^2}\frac{dR}{dv}\right\}\left(\frac{d\phi}{d\tau}-\frac{3}{4}\frac{d\phi^2}{d\tau^2}\right)+$$
$$\frac{\epsilon n}{h_{,}}\frac{d}{d\tau}\left\{\rho_{,}^2\left(\frac{d\phi}{d\tau}-\frac{d\phi^2}{d\tau^2}\right)\right\}.$$

Eliminate $(\rho_{,}\cos\lambda_{,}M+\rho_{,}\sin\lambda_{,}N)\frac{h^{\frac{1}{2}}}{2h_{,}^{\frac{1}{2}}}\left(\frac{d\phi}{d\tau}-\frac{3}{4}\frac{d\phi^2}{d\tau^2}\right)$ by

$$(S)=\left(\frac{1}{h}+\frac{2\mu\rho_{,}}{h_{,}^{\frac{1}{2}}h^{\frac{5}{2}}}\right)\frac{dR}{dv}-\frac{h^{\frac{1}{2}}}{h_{,}^{\frac{1}{2}}}(\rho_{,}\cos\lambda_{,}M+\rho_{,}\sin\lambda_{,}N)-\frac{\epsilon n}{h_{,}}\frac{d(\rho_{,}^2)}{d\tau}.$$

and we find as the result,

$$S=(S)-\left\{(S)-\frac{1}{h}\frac{dR}{dv}+\frac{\epsilon n}{h_{,}}\frac{d(\rho_{,}^2)}{d\tau}\right\}\left(\frac{1}{2}\frac{d\phi}{d\tau}-\frac{3}{8}\frac{d\phi^2}{d\tau^2}\right)+$$
$$\frac{\epsilon n}{h_{,}}\frac{d}{d\tau}\left\{\rho_{,}^2\left(\frac{d\phi}{d\tau}-\frac{d\phi^2}{d\tau^2}\right)\right\}.$$

Make $(P)=(S)-\frac{1}{h}\frac{dR}{dv}+\frac{\epsilon n}{h_{,}}\frac{d(\rho_{,}^2)}{d\tau}$. Then,

$$S=(S)-P\left(\frac{1}{2}\frac{d\phi}{d\tau}-\frac{3}{8}\frac{d\phi^2}{d\tau^2}\right)+\frac{\epsilon n}{h_{,}}\frac{d}{d\tau}\left\{\rho_{,}^2\left(\frac{d\phi}{d\tau}-\frac{d\phi^2}{d\tau^2}\right)\right\}.$$

We now develope $\rho_{,}$ and $\lambda_{,}$ by the powers of ϕ; so that (S^0) and (P^0) will have only $\rho_{,}^0$ and $\lambda_{,}^0$, which are elliptic functions of τ only, with the constant elements $a_{,}$, $e_{,}$, &c.

$$(S)=(S^0)+\phi\frac{d(S^0)}{d\tau}+\frac{\phi^2}{2}\frac{d^2(S^0)}{d\tau^2};$$

$$(P)=(P^0)+\phi\frac{d(P^0)}{d\tau}\ ;\ \rho_{,}^2=\rho_{,}^{02}+\phi\frac{d(\rho_{,}^{02})}{d\tau}.$$

And by substitution,

$$S=(S^0)+\phi\frac{d(S^0)}{d\tau}-\tfrac{1}{2}(P^0)\frac{d\phi}{d\tau}+\frac{\mathcal{E}n}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\frac{d\phi}{d\tau}+\frac{\mathcal{E}n\rho_{,}^{02}}{h_{,}}\frac{d^2\phi}{d\tau^2}\left(1-2\frac{d\phi}{d\tau}\right)$$

$$+\frac{\phi^2}{2}\frac{d^2(S^0)}{d\tau^2}+\tfrac{3}{8}(P^0)\frac{d\phi^2}{d\tau^2}-\frac{\phi}{2}(P^0)\frac{d(P^0)}{d\tau}+\frac{\mathcal{E}n}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}\phi\frac{d^2\phi}{d\tau^2}+\frac{\mathcal{E}n}{h_{,}}\frac{d^2(\rho_{,}^{02})}{d\tau^2}\phi\frac{d\phi}{d\tau}.$$

To this we must add,

$$(P^0)=(S^0)-\frac{1}{h}\frac{dR}{dv}+\frac{\mathcal{E}n}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}-\frac{\mathcal{E}n}{h_{,}}\frac{d^2(\rho_{,}^{02})}{d\tau^2}\phi,$$

$$(S^0)=(P^0)+\frac{1}{h}\frac{dR}{dv}-\frac{\mathcal{E}n}{h_{,}}\frac{d(\rho_{,}^{02})}{d\tau}+\frac{\mathcal{E}n}{h_{,}}\frac{d^2(\rho_{,}^{02})}{d\tau^2}\phi,$$

5. In concluding this paper, I will give a brief sketch of a transformation of the differential equations which determine the place of a disturbed planet, which I have never seen noticed, but which I think is deserving of some consideration.

Make $Q=-\frac{\mu}{r}+R$; the equations with reference to a fixed plane are,

$$\frac{d^2x}{dt^2}+\frac{dQ}{dx}=0,\quad \frac{d^2y}{dt^2}+\frac{dQ}{dy}=0,\quad \frac{d^2z}{dt^2}+\frac{dQ}{dz}=0. \qquad (1)$$

Suppose now a plane always to make the constant angle i with the fixed plane, i being the mean inclination of the orbit to this plane. And let this plane slide on the fixed plane in such a manner that their intersection may have a uniform motion equal to that of the node, or so that the intersection of the two planes may be always the mean place of the node. And let $\theta-ant$ be the longitude of this intersection, θ being constant.

Let now $v_{,}$ and $y_{,}$ be co-ordinates making the angle $\theta-ant$ with x and y, $x_{,}$ lying in the intersection of the sliding and fixed planes. We have

$$x = x_{,} \cos(\theta - ant) - y_{,} \sin(\theta - ant),$$
$$y = x_{,} \sin(\theta - ant) + y_{,} \cos(\theta - ant).$$

We form the two equations,

$$\cos(\theta - ant)\left(\frac{d^2x}{dt^2} + \frac{dQ}{dx}\right) + \sin(\theta - ant)\left(\frac{d^2y}{dt^2} + \frac{dQ}{dy}\right) = 0.$$
$$-\sin(\theta - ant)\left(\frac{d^2x}{dt^2} + \frac{dQ}{dx}\right) + \cos(\theta - ant)\left(\frac{d^2y}{dt^2} + \frac{dQ}{dy}\right) = 0.$$

But

$$\cos(\theta - ant)\frac{dQ}{dx} + \sin(\theta - ant)\frac{dQ}{dy} = \frac{dQ}{dx_{,}},$$
$$-\sin(\theta - ant)\frac{dQ}{dx} + \cos(\theta - ant)\frac{dQ}{dy} = \frac{dQ}{dy_{,}}.$$

Therefore,

$$\cos(\theta - ant)\frac{d^2x}{dt^2} + \sin(\theta - ant)\frac{d^2y}{dt^2} + \frac{dQ}{dx_{,}} = 0.$$
$$-\sin(\theta - ant)\frac{d^2x}{dt^2} + \cos(\theta - ant)\frac{d^2y}{dt^2} + \frac{dQ}{dy_{,}} = 0.$$

Substituting for $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$, their values in $x_{,}$ and $y_{,}$, we have

$$\frac{d^2x_{,}}{dt^2} + \frac{dQ}{dx_{,}} + 2an\frac{dy_{,}}{dt} - a^2n^2x_{,} = 0,$$
$$\frac{d^2y_{,}}{dt^2} + \frac{dQ}{dy_{,}} - 2an\frac{dx_{,}}{dt} - a^2n^2y_{,} = 0.$$

Or dropping the distinctive marks of $x_{,}$ and $y_{,}$, as tending to produce confusion, and as no longer necessary:

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} + \frac{dQ}{dx} + 2an\frac{dy}{dt} - a^2n^2x = 0, \\ &\frac{d^2y}{dt^2} + \frac{dQ}{dy} - 2an\frac{dx}{dt} - a^2n^2y = 0. \end{aligned}\right\} \quad (2)$$

We now take the new co-ordinates $y_{,}$ and $z_{,}$, $y_{,}$ lying in the sliding plane, and $z_{,}$ being perpendicular to it; and, consequently, we have

$$y = y_{\prime} \cos i - z_{\prime} \sin i, \quad z = y_{\prime} \sin i + z_{\prime} \cos i.$$

Proceeding by exactly the same steps as before, we find

$$\frac{d^2 y_{\prime}}{dt^2} + \frac{dQ}{dy_{\prime}} - 2an \cos i \cdot \frac{dx}{dt} - a^2 n^2 \cos i \,(y_{\prime} \cos i - z_{\prime} \sin i) = 0,$$

$$\frac{d^2 z_{\prime}}{dt^2} + \frac{dQ}{dz_{\prime}} + 2an \sin i \cdot \frac{dx}{dt} + a^2 n^2 \sin i \,(y_{\prime} \cos i - z_{\prime} \sin i) = 0.$$

Or again, dropping the distinctive marks of $y_{\prime}$ and $z_{\prime}$

$$\left.\begin{aligned} &\frac{d^2 y}{dt^2} + \frac{dQ}{dy} - 2an \cos i \cdot \frac{dx}{dt} - a^2 n^2 \cos i \,(y \cos i - z \sin i) = 0. \\ &\frac{d^2 z}{dt^2} + \frac{dQ}{dz} + 2an \sin i \cdot \frac{dx}{dt} + a^2 n^2 \sin i \,(y \cos i - z \sin i) = 0. \end{aligned}\right\} \quad (3)$$

Again, $x_{\prime}$ and $y_{\prime}$ lying in the sliding plane, let $x_{\prime}$ be the angle $\pi + \epsilon nt$ in advance of the line of intersection of the two planes. Then ϵ may be so determined, that $x_{\prime}$ shall lie in the mean place of the apse when projected on the sliding plane, or in any other position we please. And as before,

$$x = x_{\prime} \cos (\pi + \epsilon nt) - y_{\prime} \sin (\pi + \epsilon nt),$$

$$y = x_{\prime} \sin (\pi + \epsilon nt) + y_{\prime} \cos (\pi + \epsilon nt).$$

Making for a moment, in order to abridge,

$$A = 2an \frac{dy}{dt} - a^2 n^2 x, \quad B = -2an \cos i \cdot \frac{dx}{dt} - a^2 n^2 \cos i \,(y \cos i - z \sin i);$$

we shall have from the first of (2) and the first of (3), by exactly the same process as in the two former transformations,

$$\cos (\pi + \epsilon nt) \frac{d^2 x}{dt^2} + \sin (\pi + \epsilon nt) \frac{d^2 y}{dt^2} + \frac{dQ}{dx_{\prime}} + A \cos (\pi + \epsilon nt) + B \sin (\pi + \epsilon nt) = 0.$$

$$-\sin (\pi + \epsilon nt) \frac{d^2 x}{dt^2} + \cos (\pi + \epsilon nt) \frac{d^2 y}{dt^2} + \frac{dQ}{dy_{\prime}} - A \sin (\pi + \epsilon nt) + B \cos (\pi + \epsilon nt) = 0.$$

Or, putting for $\frac{d^2 x}{dt^2}$ and $\frac{d^2 y}{dt^2}$ their values,

$$\frac{d^2x_{,}}{dt^2}+\frac{dQ}{dx_{,}}+L_{,}=0,$$

$$\frac{d^2y_{,}}{dt^2}+\frac{dQ}{dy_{,}}+M_{,}=0.$$

And again dropping the distinctive marks,

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{dQ}{dx}+L=0,\\ \frac{d^2y}{dt^2}+\frac{dQ}{dy}+M=0.\end{aligned}\right\}\qquad(4)$$

$$L=\frac{dy}{dt}\left\{2an\cos^2\frac{i}{2}-2\epsilon n+2an\sin^2\frac{i}{2}\cos 2(\pi+\epsilon nt)\right\}+2an\frac{dx}{dt}\sin^2\frac{i}{2}\sin 2(\pi+\epsilon nt)-$$

$$x\left\{a^2n^2+\epsilon^2n^2-2a\epsilon n^2\cos^2\frac{i}{2}-\tfrac{1}{2}a^2n^2\sin^2 i+\left(\tfrac{1}{2}a^2n^2\sin^2 i-2a\epsilon n^2\sin^2\frac{i}{2}\right)\cos 2(\pi+\epsilon nt)\right\}+$$

$$y\left(\tfrac{1}{2}a^2n^2\sin^2 i-2a\epsilon n^2\sin^2\frac{i}{2}\right)\sin 2(\pi+\epsilon nt)+a^2n^2\sin i\cos iz\sin(\pi+\epsilon nt).$$

$$N=\frac{dx}{dt}\left\{2\epsilon n-2an\cos^2\frac{i}{2}+2an\sin^2\frac{i}{2}\cos 2(\pi+\epsilon nt)\right\}-2an\frac{dy}{dt}\sin^2\frac{i}{2}\sin 2(\pi+\epsilon nt)+$$

$$y\left\{2a\epsilon n^2\cos^2\frac{i}{2}-a^2n^2-\epsilon^2n^2+\tfrac{1}{2}a^2n^2\sin^2 i+\left(\tfrac{1}{2}a^2n^2\sin^2 i-2a\epsilon n^2\sin^2\frac{i}{2}\right)\cos 2(\pi+\epsilon nt)\right\}+$$

$$x\left(\tfrac{1}{2}a^2n^2\sin^2 i-2a\epsilon n^2\sin^2\frac{i}{2}\right)\sin 2(\pi+\epsilon nt)+a^2n^2\sin i\cos iz\cos(\pi+\epsilon nt).$$

The terms in L and M containing z are of the order of the third power of the disturbing force multiplied by $\sin i$, and will not be wanted in these cases in which i is large, and, I think, cannot be wanted when it is small. Most of the other terms are of the second order; and as i is constant, they are all of a very simple character.

We must add from the last of (3)

$$\frac{d^2z}{dt^2}+\frac{dQ}{dz}+N=0.\qquad(5)$$

$$N=2an\sin i\frac{dx}{dt}+a^2n^2\sin i\,(y\cos i-z\sin i).$$

The term in N containing z is of the order of the third power of the disturbing force multiplied by $\sin^2 i$, and cannot, I conceive, in any case be wanted.

We may transform the co-ordinates x', y', and z', of the disturbing body in like manner to a sliding plane, and to a similar situation on that plane. Thus, marking all the corresponding quantities relative to this body with an accent, we have

$$x' = x'_1 \cos(\theta' - \alpha'n't) - y'_1 \sin(\theta' - \alpha'n't),$$

$$y' = x'_1 \sin(\theta' - \alpha'n't) + y'_1 \cos(\theta' - \alpha'n't), \; z' = z'_1.$$

$$x'_1 = x'_2, \; y'_1 = y'_2 \cos i' - z'_2 \sin i', \; z'_1 = y'_2 \sin i' - z'_2 \cos i'.$$

$$x'_2 = x'_3 \cos(\pi' + \epsilon'n't) - y'_3 \sin(\pi' + \epsilon'n't),$$

$$y'_2 = x'_3 \sin(\pi' + \epsilon'n't) + y'_3 \cos(\pi' + \epsilon'n't), \; z'_2 = z'_3,$$

But we may transform x', y', and z' to the plane of the orbit of the disturbing body, if we please.

To find the latitude, marking the letters in the second member with the number of the transformation to which they belong, we have

$$z = y_2 \sin i + z_2 \cos i = x_3 \sin i \sin(\pi + \epsilon nt) + y_3 \sin i \cos(\pi + \epsilon nt) + z_3 \cos i.$$

But σ being the sine of the true latitude, s the sine of the latitude relative to the sliding plane, and f the anomaly, or the longitude measured from the axis of x after the last transformation;

$$z = r\sigma, \; x_3 = r\cos f, \; y_3 = r\sin f, \text{ and } z_3 = rs.$$

Therefore, by substitution in the last, and dividing by r,

$$\sigma = \sin i \{\sin(\pi + \epsilon nt)\cos f + \cos(\pi + \epsilon nt)\sin f\} + s\cos i;$$

or,

$$\sigma = \sin i \sin(f + \pi + \epsilon nt) + s\cos i. \tag{6}$$

Thus the latitude is very easily found.

Still marking the letters according to the transformation to which they belong,

$$y_1 = y_2 \cos i - z_2 \sin i = x_3 \cos i \sin(\pi + \epsilon nt) + y_3 \cos i \cos(\pi + \epsilon nt) - z_3 \sin i.$$

Let v be the longitude on the fixed plane, measured from its intersection with the sliding plane,

$$y_1 = r \sin v,$$

the rest as before ; then, by substitution,

$$\sin v = \cos i \{\cos f \sin (\pi + \epsilon nt) + \sin f \cos (\pi + \epsilon nt)\} - s \sin i.$$

Or,

$$\sin v = \cos i \sin (f + \pi + \epsilon nt) - s \sin i. \qquad (7)$$

This will give the longitude when i is large ; and when it is small, we can without difficulty find v itself without finding its sine.

I must, in conclusion, repeat what I have before said, that I think this transformation worthy of consideration.

GUNTHWAITE HALL,
NEAR BARNSLEY, YORKSHIRE,
Nov. 13, 1848.